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Homogenization of time harmonic Maxwell equations and the frequency dispersion effect

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Abstract

We perform homogenization of the time-harmonic Maxwell equations in order to determine the effective dielectric permittivity ε^h and effective electric conductivity σ^h. We prove that ε^h and σ^h depend on the pulsation ω; this phenomenon is known as the frequency dispersion effect. Moreover, the macroscopic Maxwell equations also depend on ω; they are different for small and large values of ω.

Résumé

On étudie l'homogénéisation des équations de Maxwell en régime harmonique, afin de déterminer la permittivité diélectrique effective ε^h et la conductivité électrique effective σ^h. On montre que ε^h et σ^h dépendent de la pulsation ω ; ce phénomène est connu sous le nom d'effet de dispersion de fréquence. En outre, les équations de Maxwell macroscopiques dépendent également de ω ; elles sont différentes pour les petites et les grandes valeurs de ω.

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1. Introduction

The theory of effective parameters of electro-composite materials has a long history [37] starting with the paper by Lorenz [24]. There is a number of phenomenological mixing formulae which allow to calculate the effective dielectric permittivity and effective electric conductivity from a given composition of the composite. The best known mixing formulae are given by Garnett and von Bruggeman [53].

The two-scale homogenization approach is one more tool in finding the effective parameters. The approach is well justified mathematically for a composite medium with heterogeneous microstructure which is described by spatially

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periodic parameters when the composite medium is subject to electromagnetic fields generated by currents of varying frequencies [4,22,31,51].

For general time-dependent solutions, it was proved in many studies by the two-scale homogenization approach that the macroscopic Maxwell equations can be strongly different from the microscopic ones: instantaneous material laws turn into constitutive laws with memory [5,12,18,21,25,34,38,39,49,50]. A more general case has been considered in [7] with polarization of composite ingredients being not instantaneous but obeying the Debye or Lorenz polarization laws with relaxation. The complexity of the macroscopic constitutive laws is discussed in [10].

The present study is motivated by geophysical applications such as the oil-reservoir logging [19]. There is an evidence (see [41] for example) that the electric conductivity and dielectric permittivity of real rocks can depend strongly on the frequency ω of the current source. This is why we focus on the frequency dispersion effect while performing the two-scale homogenization of the Maxwell equations. The dispersion effect has been verified numerically, see the recent papers [35,36]. Here, we perform a mathematical justification for the underlying boundary-value problems resulting from the homogenization analysis. The dispersion effect considered in the present paper is due to the Maxwell–Wagner mechanism: free charges concentrate on interphase surfaces to provide continuity of electric currents across such surfaces [26,48]. This is why the resulting polarization of the mixture depends on the pulsation of the current source. When passing to clay-containing rocks one should also take into account bound charges concentrating on the interface surfaces. Such rocks are not considered here.

In papers on two-scale homogenization of the Maxwell equations, it is assumed commonly that the wave length l_w is much greater than the size l of the representative periodicity cell of the heterogeneous medium, $\delta = l/L$ being a small dimensionless parameter, where L is a size of the domain of measurements Ω. It is a novelty of the present study that in performing homogenization we take into account not only the wave length l_w but the skin layer length l_s as well; it is another intrinsic length parameter characterizing a composite material in the presence of a time-harmonic electric field source. Up to now, the skin layer length was disregarded in all the mathematical publications and it have given grounds to physicists to discuss limitations of the two-scale homogenization approach [46]. Observe that both l_w and l_s depend on the pulsation ω. Hence, the order relation between the lengths l, l_w, and l_s depends also on ω; it may occur that $l \ll l_w \ll l_s$ or $l \ll l_s \ll l_w$ and etc. We systematically analyze all the possible order relations which are of interest in geophysics (see hypotheses (7), (9), and (10)).

The two-scale homogenization is a well established method in the theory of partial differential equations with rapidly oscillating periodic coefficients. This method has a lot of important applications in various branches of physics, mechanics and modern technology: porous media, composite and perforated materials, thermal conduction, acoustics, electromagnetism.

For general references on the homogenization theory we refer to [1,2,6,8,9,13–15,17,27–30,34,40,42–45,52].

The paper is organized as follows. In Section 2 we consider the time harmonic Maxwell equations with periodic varying coefficients depending on δ. To give due consideration to different order relations between the lengths l, l_w, and l_s, a nondimensionalization is done. We derive equations depending on δ, each one corresponding to a class frequencies, leading to homogenization and singular perturbation problems. In Section 3 we derive energy equalities and in Section 4 we show the unique solvability of the problems under consideration. Section 5 is devoted to homogenization. We successively consider the low frequency case, high and very high frequency cases, quasi-stationary and low conductivity cases, and the medium frequency case. We find the macroscopic equations resulting from homogenization of the Maxwell equations. We derive boundary-value problems which allow for determination of the effective constant matrices μ^h, ε^h, and σ^h; the effective matrices and the macroscopic equations depend on the pulsation ω. Section 6 provides a conclusion where we emphasize the principal novelties of the present study.

2. Problem formulation

In what follows, we use the Gaussian system of units. Given a current density $J_s \equiv e^{-i\omega t}\mathbf{f}(x)$ with a pulsation ω, the electromagnetic fields $E \equiv e^{-i\omega t}\mathbf{E}(x)$, $D \equiv e^{-i\omega t}\mathbf{D}(x)$, $H \equiv e^{-i\omega t}\mathbf{H}(x)$, $B \equiv e^{-i\omega t}\mathbf{B}(x)$, $J \equiv e^{-i\omega t}\mathbf{J}(x)$ solve the time-harmonic Maxwell equations:

$$-\frac{i\omega}{c}\mathbf{D} = \operatorname{curl}\mathbf{H} - \frac{4\pi}{c}\mathbf{J} - \frac{4\pi}{c}\mathbf{f}, \qquad \frac{i\omega}{c}\mathbf{B} = \operatorname{curl}\mathbf{E}, \tag{1}$$

with the material laws,

$$\mathbf{D} = \varepsilon(x)\mathbf{E}, \qquad \mathbf{B} = \mu(x)\mathbf{H}, \qquad \mathbf{J} = \sigma(x)\mathbf{E}. \tag{2}$$

Here, ε is the dielectric permittivity, σ is the electric conductivity, and μ is the magnetic permeability. By cross differentiation, we exclude the magnetic field to switch to the Helmholtz-like equation:

$$\operatorname{curl}\left(\frac{1}{\mu}\operatorname{curl}\mathbf{E}\right) = \kappa^2\mathbf{E} + \frac{i4\pi\omega}{c^2}\mathbf{f}, \quad \kappa^2 = \frac{\omega^2\varepsilon + i4\pi\sigma\omega}{c^2}. \tag{3}$$

The components of the mixture are assumed to be isotropic. The periodic rock structure implies that the functions ε, σ, and μ in (2) are periodic, i.e.

$$\varepsilon(x_1 + l_1, x_2, x_3) = \varepsilon(x_1, x_2 + l_2, x_3) = \varepsilon(x_1, x_2, x_3 + l_3) = \varepsilon(x_1, x_2, x_3)$$

for any x, and σ, μ satisfy a similar property. Given a size L of a bounded domain of measurements $\Omega \subset \mathbb{R}^3$, we use the small ratios,

$$\frac{l_j}{L} = r_j\delta, \quad \min\{r_1, r_2, r_3\} = 1,$$

where δ is a small dimensionless parameter. The dimensionless parameters r_j characterize the deviation of the representative cell of periodicity $Y^\delta = \prod_{j=1}^3 (0, l_j)$ from a regular cube. Particularly, $r_1 = r_2 = r_3 = 1$ if all the sizes l_j are equal. For simplicity, we consider a composite material with two different constituents, say fluid and solid. Let Y_f^δ and Y_s^δ be the subdomains of Y^δ occupied by the fluid and the solid, respectively, $Y_f^\delta \cup Y_s^\delta = Y^\delta$, and let Γ^δ be the interface between the solid and fluid parts. The coefficients in Eq. (3) are discontinuous step functions; their restrictions to the representative cell Y^δ are given by:

$$\varepsilon(x), \mu(x), \sigma(x) = \begin{cases} \varepsilon_s, \mu_s, \sigma_s, & \text{if } x \in Y_s^\delta, \\ \varepsilon_f, \mu_f, \sigma_f, & \text{if } x \in Y_f^\delta. \end{cases}$$

Tangential components of electric and magnetic fields are continuous across the interface Γ^δ [20]:

$$[\mathbf{n} \wedge \mathbf{E}] = 0, \qquad \left[\mathbf{n} \wedge \mu^{-1}\operatorname{curl}\mathbf{E}\right] = 0. \tag{4}$$

Here, "$\wedge$" stands for the vector product, $\mathbf{n}$ is the unit normal vector to Γ^δ, and the brackets $[\cdot]$ stand for the jump across Γ^δ.

Let Ω be a bounded smooth domain in $\mathbb{R}^3$. Given $m_i \in \mathbb{Z}$, we define a shift of Y^δ as

$$Y_m^\delta = \{x\colon\ 0 < x_i - m_i < l_i\}, \quad m = (m_1, m_2, m_3),$$

and similarly we define the shifts Y_{fm}^δ and Y_{sm}^δ. We introduce the fluid and solid domains:

$$\Omega_f^\delta = \Omega \cap \left(\bigcup_{m\in\mathbb{Z}^3} Y_{fm}^\delta\right), \qquad \Omega_s^\delta = \Omega \cap \left(\bigcup_{m\in\mathbb{Z}^3} Y_{sm}^\delta\right).$$

Eq. (3) holds in the domains Ω_f^δ and Ω_s^δ, and we impose the perfect conductor boundary condition on the boundary $\partial\Omega$:

$$\mathbf{E} \wedge \mathbf{n} = 0, \tag{5}$$

$\mathbf{n}$ being the unit outward normal vector to $\partial\Omega$.

Our objective is to perform an asymptotic analysis of problem (3)–(5), as $\delta \to 0$.

With $\hat{\mathbf{E}}$ standing for the reference value of $\mathbf{E}$, we introduce the dimensionless variables $x_i' = x_i/L$, and

$$\mathbf{E}' = \frac{\mathbf{E}}{\hat{E}}, \qquad \mathbf{D}' = \frac{\mathbf{D}}{\hat{E}}, \qquad \mathbf{H}' = \frac{\mathbf{H}}{\hat{H}}, \qquad \mathbf{B}' = \frac{\mathbf{B}}{\hat{H}}, \qquad \mathbf{J}' = \frac{\mathbf{J}}{\hat{J}}, \qquad \mathbf{f}' = \frac{\mathbf{f}}{\hat{J}}, \qquad \omega' = \frac{\omega}{\hat{\omega}}, \qquad \sigma' = \frac{\sigma}{\hat{\sigma}}.$$

Let $\hat{\varepsilon}$ and $\hat{\mu}$ be reference values of ε and μ, $\varepsilon = \hat{\varepsilon}\varepsilon'$, $\mu = \hat{\mu}\mu'$. We remind that both ε and μ are dimensionless parameters in the Gaussian system of units. We choose $\hat{J} = \hat{\sigma}\hat{E}$ then, in the dimensionless variables, Eq. (3) becomes:

$$\operatorname{curl}'\left(\frac{1}{\mu'}\operatorname{curl}'\mathbf{E}'\right)-\kappa'^2\mathbf{E}'=ia_1 4\pi\omega'\mathbf{f}', \tag{6}$$

with

$$\kappa'^2=\omega'^2\varepsilon'\frac{L^2}{l_w^2}+i4\pi\sigma'\omega'\frac{L^2}{l_s^2},\qquad a_1=\frac{L^2}{l_s^2},\qquad l_w=\frac{c}{\hat\omega\sqrt{\hat\mu\hat\varepsilon}},\qquad l_s=\frac{c}{\sqrt{\hat\omega\hat\sigma\hat\mu}}.$$

Here, l_w is the wave length and l_s is the skin layer length. We use the dimensionless microscopic variables:

$$y_j=\frac{x'_j}{\delta},\quad y\in Y=\prod_{j=1}^{3}(0,r_j).$$

Here, Y is the dimensionless periodicity cell consisting of solid and fluid parts, $Y=Y_s\cup Y_f$, and we have:

$$\varepsilon'(y),\mu'(y),\sigma'(y)=\begin{cases}\varepsilon'_s,\mu'_s,\sigma'_s, & \text{if } y\in Y_s,\\ \varepsilon'_f,\mu'_f,\sigma'_f, & \text{if } y\in Y_f.\end{cases}$$

The assumption on the periodic rock structure implies that the coefficients ε', σ', and μ' in Eq. (6) are periodic step functions:

$$\varepsilon'\left(\frac{x'}{\delta}\right)=\varepsilon'_\delta(x'),\qquad \mu'\left(\frac{x'}{\delta}\right)=\mu'_\delta(x'),\qquad \sigma'\left(\frac{x'}{\delta}\right)=\sigma'_\delta(x'),$$

with the period δr_j in each variable x'_j.

Low, medium, and high frequency cases. Given $q\geqslant 0$, we consider the hypothesis:

$$\frac{l_j}{L}=r_j\delta,\qquad \frac{l_j}{l_s}=\alpha_s^j\delta^{1-q},\qquad \frac{l_j}{l_w}=\alpha_w^j\delta^{1-q}. \tag{7}$$

The value $q=0$ corresponds to the case of low angular frequency $\hat\omega$, when the skin layer length and the wave length are greater than the cell size. The values $q\geqslant 1$ correspond to high frequencies, and the values $0<q<1$ correspond to medium frequencies.

For simplicity, we assume that the ratios α_s^j/r_j and α_w^j/r_j are independent of the index j. Under hypothesis (7), we have:

$$\frac{L}{l_s}=\frac{\alpha_s^1}{r_1\delta^q}\equiv\frac{\alpha_s}{\delta^q},\qquad \frac{L}{l_w}=\frac{\alpha_w^1}{r_1\delta^q}\equiv\frac{\alpha_w}{\delta^q},\qquad a_1=\frac{\alpha_s^2}{\delta^{2q}},$$

then

$$\kappa_\delta'^2=\frac{(k'_{1\delta})^2}{\delta^{2q}},\qquad (k'_{1\delta})^2(x')=\alpha_w^2\omega'^2\varepsilon'_\delta(x')+i4\pi\alpha_s^2\omega'\sigma'_\delta(x'),$$

where the amplitude of $\kappa_{1\delta}'^2$ does not depend on δ. Eq. (6) becomes:

$$\operatorname{curl}'\left(\frac{1}{\mu'_\delta}\operatorname{curl}'\mathbf{E}'\right)-\frac{(k'_{1\delta})^2}{\delta^{2q}}\mathbf{E}'=\frac{i4\pi\omega'\alpha_s^2}{\delta^{2q}}\mathbf{f}'. \tag{8}$$

Quasi-stationary case. At high frequencies, when one of the medium component is highly conductive and dielectric permittivity of both the components is low, it may occur that the wave length is much greater than the pore size but the skin layer length and the pore diameter are of the same size:

$$\frac{l_j}{L}=\delta r_j,\qquad \frac{l_j}{l_w}=\delta\alpha_w^j,\qquad \frac{l_j}{l_s}=\alpha_s^j. \tag{9}$$

As above, we assume that the ratios α_s^j/r_j and α_w^j/r_j are independent of the index j. We denote $\alpha_s^1/r_1=\alpha_s$ and $\alpha_w^1/r_1=\alpha_w$. In this case $a_1=(L/l_s)^2=\alpha_s^2\delta^{-2}$, and Eq. (6) becomes:

$$\delta^2 \operatorname{curl}'\biggl(\frac{1}{\mu'_\delta}\operatorname{curl}'\mathbf{E}'\biggr) - \bigl(\delta^2\omega'^2\varepsilon'_\delta\alpha_w^2 + i4\pi\omega'\sigma'_\delta\alpha_s^2\bigr)\mathbf{E}' = i4\pi\omega'\alpha_s^2\mathbf{f}'.$$

Low conductivity case. At high frequencies, when the electric conductivity of both the components is low, it may occur that

$$\frac{l_j}{L} = \delta r_j, \qquad \frac{l_j}{l_w} = \alpha_w^j, \qquad \frac{l_j}{l_s} = \delta\alpha_s^j. \tag{10}$$

Assuming that the ratios α_s^j/r_j and α_w^j/r_j are independent of the index j, and denoting $\alpha_s^1/r_1 = \alpha_s$ and $\alpha_w^1/r_1 = \alpha_w$, we obtain $a_1 = (L/l_s)^2 = \alpha_s^2$, and Eq. (6) becomes:

$$\delta^2 \operatorname{curl}'\biggl(\frac{1}{\mu'_\delta}\operatorname{curl}'\mathbf{E}'\biggr) - \bigl(\omega'^2\varepsilon'_\delta\alpha_w^2 + i4\pi\delta^2\omega'\sigma'_\delta\alpha_s^2\bigr)\mathbf{E}' = i4\pi\delta^2\omega'\alpha_s^2\mathbf{f}'.$$

3. Energy equalities

Let us introduce the Hilbert space:

$$H(\operatorname{curl}, \Omega) = \bigl\{\mathbf{u} \in L^2\bigl(\Omega; \mathbb{C}^3\bigr),\ \operatorname{curl}\mathbf{u} \in L^2\bigl(\Omega; \mathbb{C}^3\bigr)\bigr\},$$

with the scalar product

$$(\mathbf{u}, \mathbf{v}) = \int_\Omega \mathbf{u}\cdot\mathbf{v}^* + \operatorname{curl}\mathbf{u}\cdot\operatorname{curl}\mathbf{v}^*\,dx,$$

where $\mathbf{v}^*$ stands for the complex conjugate of $\mathbf{v}$ and "$\cdot$" for the scalar product in $\mathbb{R}^3$.

In the sequel we assume that Ω is a smooth bounded domain. The tangential trace γ_τ on $\partial\Omega$ is defined on $H(\operatorname{curl}, \Omega)$ by $\gamma_\tau\mathbf{u} = (\mathbf{n}\wedge\mathbf{u})|_{\partial\Omega}$ and it is well known that $(\mathbf{n}\wedge\mathbf{u})|_{\partial\Omega}$ belongs to $H^{-1/2}(\partial\Omega; \mathbb{C}^3)$. We denote:

$$H_0(\operatorname{curl}, \Omega) = \bigl\{\mathbf{u} \in H(\operatorname{curl}, \Omega),\ (\mathbf{n}\wedge\mathbf{u})|_{\partial\Omega} = 0\bigr\}.$$

The space $H_0(\operatorname{curl}, \Omega)$ can be obtained as the closure of $\mathcal{C}_0^\infty(\Omega)$ in $H(\operatorname{curl}, \Omega)$, see e.g. [16] (p. 240). Notice that the tangential trace $(\mathbf{n}\wedge\mathbf{u})|_{\partial\Omega}$ belongs to a proper subspace of $H^{-1/2}(\partial\Omega; \mathbb{C}^3)$. To specify the result [3] we introduce the tangential divergence of a vector field $\mathbf{u} \in H^{-1/2}(\partial\Omega; \mathbb{C}^3)$ by:

$$\langle \operatorname{div}_{\partial\Omega}\mathbf{u}, v\rangle_{\partial\Omega} = -\bigl\langle \mathbf{u}, (\nabla v)|_{\partial\Omega}\bigr\rangle_{\partial\Omega}, \quad \forall v \in H^{3/2}(\partial\Omega; \mathbb{C}),$$

where $\langle\cdot,\cdot\rangle$ stands in the left for the duality pairing between $H^{-3/2}(\partial\Omega; \mathbb{C})$, and $H^{3/2}(\partial\Omega; \mathbb{C})$, and for the duality pairing between the spaces $H^{-1/2}(\partial\Omega; \mathbb{C}^3)$, and $H^{1/2}(\partial\Omega; \mathbb{C}^3)$ in the right. Let us now introduce the Hilbert space:

$$H^{-1/2}(\operatorname{div}, \partial\Omega) = \bigl\{\mathbf{u} \in H^{-1/2}\bigl(\partial\Omega; \mathbb{C}^3\bigr),\ (\mathbf{u}\cdot\mathbf{n})|_{\partial\Omega} = 0,\ \operatorname{div}_{\partial\Omega}\mathbf{u} \in H^{-1/2}(\partial\Omega; \mathbb{C})\bigr\}$$

equipped with the norm,

$$\|\mathbf{u}\|_{H^{-1/2}(\operatorname{div},\partial\Omega)} = \|\mathbf{u}\|_{H^{-1/2}(\partial\Omega;\mathbb{C}^3)} + \|\operatorname{div}_{\partial\Omega}\mathbf{u}\|_{H^{-1/2}(\partial\Omega;\mathbb{C})}.$$

Then γ_τ is a linear continuous operator from $H(\operatorname{curl}, \Omega)$ into $H^{-1/2}(\operatorname{div}, \partial\Omega)$ and is surjective. We refer to [11] for an other characterization of tangential traces of $H(\operatorname{curl}, \Omega)$.

Now we introduce the definition of a weak solution of problem (4)–(6) and derive the energy equalities, considering the different frequency cases. For simplicity of notation, we shall now drop the prime superscript.

Low frequency case $q = 0$. A function $\mathbf{E}^\delta$ is a classical solution of problem (4)–(6) if

$$\mathbf{E}^\delta \in C^1\bigl(\overline{\Omega_s^\delta}\bigr) \cap C^1\bigl(\overline{\Omega_f^\delta}\bigr) \cap C^2\bigl(\Omega_s^\delta\bigr) \cap C^2\bigl(\Omega_f^\delta\bigr), \tag{11}$$

$\mathbf{E}^\delta$ solves Eq. (6) in the domains Ω_s^δ and Ω_f^δ, and satisfies conditions (4) and (5).

Let us now introduce the sesquilinear form,

$$a_\delta(\mathbf{v},\mathbf{u}) = \int_\Omega \mu_\delta^{-1} \operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\mathbf{u}^* - \kappa_\delta^2 \mathbf{v}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{v},\mathbf{u}\in H_0(\operatorname{curl},\Omega), \tag{12}$$

with

$$\kappa_\delta^2 = \alpha_w^2\omega^2\varepsilon_\delta + i4\pi\alpha_s^2\omega\sigma_\delta.$$

Clearly, any function $\mathbf{E}^\delta$ satisfying conditions (4) and (11) belongs to the space $H(\operatorname{curl},\Omega)$. This is why we say that a function $\mathbf{E}^\delta \in H_0(\operatorname{curl},\Omega)$ is a weak solution of problem (4)–(6) if it satisfies the integral identity:

$$a_\delta(\mathbf{E}^\delta,\mathbf{u}) = i4\pi\alpha_s^2\omega \int_\Omega \mathbf{f}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{u}\in H_0(\operatorname{curl},\Omega). \tag{13}$$

Denoting

$$\mathbf{E}^\delta = \mathbf{E}_1^\delta + i\mathbf{E}_2^\delta, \qquad \mathbf{f} = \mathbf{f}_1 + i\mathbf{f}_2,$$

and putting $\mathbf{u} = \mathbf{E}^\delta$ in (13), we derive the energy equalities:

$$\int_\Omega \mu_\delta^{-1}\left|\operatorname{curl}\mathbf{E}^\delta\right|^2 dx = \int_\Omega \varepsilon_\delta\omega^2\alpha_\omega^2\left|\mathbf{E}^\delta\right|^2 + 4\pi\omega\alpha_s^2\left(\mathbf{f}_1\cdot\mathbf{E}_2^\delta - \mathbf{f}_2\cdot\mathbf{E}_1^\delta\right)dx, \tag{14}$$

$$\int_\Omega \sigma_\delta\left|\mathbf{E}^\delta\right|^2 dx = -\int_\Omega \mathbf{f}_1\cdot\mathbf{E}_1^\delta + \mathbf{f}_2\cdot\mathbf{E}_2^\delta\,dx. \tag{15}$$

Remark 1. In [51] the electric conductivity μ_δ and dielectric permittivity ε_δ are matrices and they should be in agreement in order to produce a coercive form. Observe that the results of [51] concern unconductive media with $\sigma_\delta \equiv 0$ and do not cover our case when

$$\mu_{ij,\delta}(x) = \mu_{0,\delta}(x)\delta_{ij}, \qquad \varepsilon_{ij,\delta}(x) = \varepsilon_{0,\delta}(x)\delta_{ij}, \qquad \sigma_{ij,\delta}(x) = \sigma_{0,\delta}(x)\delta_{ij}, \quad \sigma_0 \neq 0.$$

Medium and high frequency cases. Under conditions (7) with $q > 0$, we say that $\mathbf{E}^\delta \in H_0(\operatorname{curl},\Omega)$ is a weak solution of problem (4)–(6) if it satisfies the integral identity,

$$a_{1\delta}(\mathbf{E}^\delta,\mathbf{u}) = i4\pi\alpha_s^2\omega \int_\Omega \mathbf{f}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{u}\in H_0(\operatorname{curl},\Omega), \tag{16}$$

where $a_{1\delta}$ is the sesquilinear form defined by:

$$a_{1\delta}(\mathbf{v},\mathbf{u}) = \int_\Omega \delta^{2q}\mu_\delta^{-1} \operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\mathbf{u}^* - \kappa_{1\delta}^2 \mathbf{v}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{v},\mathbf{u}\in H_0(\operatorname{curl},\Omega), \tag{17}$$

with

$$\kappa_{1\delta}^2 = \alpha_w^2\omega^2\varepsilon_\delta + i4\pi\alpha_s^2\omega\sigma_\delta.$$

Clearly, a weak solution $\mathbf{E}^\delta$ satisfies (15) and the equality:

$$\int_\Omega \delta^{2q}\mu_\delta^{-1}\left|\operatorname{curl}\mathbf{E}^\delta\right|^2 dx = \int_\Omega \varepsilon_\delta\omega^2\alpha_\omega^2\left|\mathbf{E}^\delta\right|^2 + 4\pi\omega\alpha_s^2\left(\mathbf{f}_1\cdot\mathbf{E}_2^\delta - \mathbf{f}_2\cdot\mathbf{E}_1^\delta\right)dx. \tag{18}$$

In what follows we shall distinguish three cases: $0 < q < 1$, $q = 1$, and $q > 1$.

Quasi-stationary case. Under conditions (9), we say that $\mathbf{E}^\delta \in H_0(\operatorname{curl},\Omega)$ is a weak solution of problem (4)–(6) if it satisfies the integral identity,

$$a_{2\delta}\big(\mathbf{E}^\delta, \mathbf{u}\big) = i4\pi\alpha_s^2\omega \int\limits_\Omega \mathbf{f}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{u}\in H_0(\operatorname{curl}, \Omega), \tag{19}$$

where $a_{2\delta}$ is the sesquilinear form defined by:

$$a_{2\delta}(\mathbf{v}, \mathbf{u}) = \int\limits_\Omega \delta^2\mu_\delta^{-1}\operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\mathbf{u}^* - \kappa_{2\delta}^2\mathbf{v}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{v}, \mathbf{u}\in H_0(\operatorname{curl}, \Omega), \tag{20}$$

with

$$\kappa_{2\delta}^2 = \delta^2\alpha_w^2\omega^2\varepsilon_\delta + i4\pi\alpha_s^2\omega\sigma_\delta.$$

Clearly, a weak solution $\mathbf{E}^\delta$ satisfies (15) and the equality:

$$\int\limits_\Omega \delta^2\mu_\delta^{-1}\big|\operatorname{curl}\mathbf{E}^\delta\big|^2\,dx = \int\limits_\Omega \delta^2\varepsilon_\delta\omega^2\alpha_\omega^2\big|\mathbf{E}^\delta\big|^2 + 4\pi\omega\alpha_s^2\big(\mathbf{f}_1\cdot\mathbf{E}_2^\delta - \mathbf{f}_2\cdot\mathbf{E}_1^\delta\big)\,dx.$$

Low conductivity case. Under conditions (10), we say that $\mathbf{E}^\delta\in H_0(\operatorname{curl}, \Omega)$ is a weak solution of problem (4)–(6) if it satisfies the integral identity,

$$a_{3\delta}\big(\mathbf{E}^\delta, \mathbf{u}\big) = i4\pi\delta^2\alpha_s^2\omega \int\limits_\Omega \mathbf{f}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{u}\in H_0(\operatorname{curl}, \Omega), \tag{21}$$

where $a_{3\delta}$ is the sesquilinear form defined by:

$$a_{3\delta}(\mathbf{v}, \mathbf{u}) = \int\limits_\Omega \delta^2\mu_\delta^{-1}\operatorname{curl}\mathbf{v}\cdot\operatorname{curl}\mathbf{u}^* - \kappa_{3\delta}^2\mathbf{v}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{v}, \mathbf{u}\in H_0(\operatorname{curl}, \Omega), \tag{22}$$

with

$$\kappa_{3\delta}^2 = \alpha_w^2\omega^2\varepsilon_\delta + i4\pi\delta^2\alpha_s^2\omega\sigma_\delta.$$

Clearly, a weak solution $\mathbf{E}^\delta$ satisfies (15) and the equality:

$$\int\limits_\Omega \delta^2\mu_\delta^{-1}\big|\operatorname{curl}\mathbf{E}^\delta\big|^2\,dx = \int\limits_\Omega \varepsilon_\delta\omega^2\alpha_\omega^2\big|\mathbf{E}^\delta\big|^2 + 4\pi\delta^2\omega\alpha_s^2\big(\mathbf{f}_1\cdot\mathbf{E}_2^\delta - \mathbf{f}_2\cdot\mathbf{E}_1^\delta\big)\,dx.$$

4. Extended system and unique solvability

Our main goal here is to show that problems (13), (16), (19) and (21) are uniquely solvable for any fixed values of the parameters $\omega > 0$, $\delta > 0$. Observe that the sesquilinear forms (12), (17), (20) and (22) are not coercive and the Lax–Milgram theorem is not applicable.

Let us consider problem (3)–(5) for any value of $\omega > 0$. In accordance with the definition of weak solution introduced in Section 3, $\mathbf{E}\in H_0(\operatorname{curl}, \Omega)$ is a weak solution of problem (3)–(5) if it satisfies the integral identity:

$$\int\limits_\Omega \mu^{-1}\operatorname{curl}\mathbf{E}\cdot\operatorname{curl}\mathbf{u}^* - \kappa^2\mathbf{E}\cdot\mathbf{u}^*\,dx = \frac{i4\pi\omega}{c^2}\int\limits_\Omega \mathbf{f}\cdot\mathbf{u}^*\,dx, \quad \forall \mathbf{u}\in H_0(\operatorname{curl}, \Omega).$$

Denoting

$$\mathbf{E} = \mathbf{E}_1 + i\mathbf{E}_2, \qquad \mathbf{f} = \mathbf{f}_1 + i\mathbf{f}_2,$$

we have the energy equalities:

$$\int\limits_\Omega \mu^{-1}|\operatorname{curl}\mathbf{E}|^2\,dx = \int\limits_\Omega \frac{\omega^2\varepsilon}{c^2}|\mathbf{E}|^2 + \frac{i4\pi\omega}{c^2}(\mathbf{f}_1\cdot\mathbf{E}_2 - \mathbf{f}_2\cdot\mathbf{E}_1)\,dx, \tag{23}$$

$$\int\limits_\Omega \sigma|\mathbf{E}|^2\,dx = -\int\limits_\Omega \mathbf{f}_1\cdot\mathbf{E}_1 + \mathbf{f}_2\cdot\mathbf{E}_2\,dx. \tag{24}$$

We assume that $\mathbf{f} \in H(\mathrm{div}, \Omega)$ and that ε, μ and σ are given functions satisfying,

$$\varepsilon, \mu, \sigma \in L^\infty(\Omega), \quad 0 < c_0 \leqslant \varepsilon, \mu, \sigma \leqslant c_0^{-1}. \tag{25}$$

The existence of a weak solution to problem (3)–(5) will be obtained in two steps. Firstly, we consider the case when ε, μ and σ are smooth functions of the variable $x \in \Omega$ and the inequalities $0 < c_0 \leqslant \varepsilon, \mu, \sigma \leqslant c_0^{-1}$ hold uniformly in $x \in \Omega$. One can verify that the function $\mathbf{E}$ is a weak solution of problem (3)–(5) if and only if $(\mathbf{E}, \mathbf{H})^T \in H_0(\mathrm{curl}, \Omega) \times H(\mathrm{curl}, \Omega)$ is a strong solution of the system:

$$\frac{ic}{\varepsilon} \operatorname{curl} \mathbf{H} - i4\pi \frac{\sigma}{\varepsilon} \mathbf{E} - \omega \mathbf{E} - i4\frac{\pi}{\varepsilon} \mathbf{f} = 0, \tag{26}$$

$$-\frac{ic}{\mu} \operatorname{curl} \mathbf{E} - \omega \mathbf{H} = 0. \tag{27}$$

The value of $\mathbf{n} \cdot \operatorname{curl} \mathbf{E}$ on the boundary $\partial\Omega$ can be expressed in local coordinates by the formula [11]:

$$\mathbf{n} \cdot \operatorname{curl} \mathbf{E} = \operatorname{div}_{\partial\Omega} \mathbf{E} \wedge \mathbf{n}|_{\partial\Omega}. \tag{28}$$

It follows from (5), (27) and (28) that the field $\mathbf{H}$ satisfies the following boundary condition on $\partial\Omega$:

$$\mathbf{n} \cdot \mathbf{H} = 0.$$

We introduce

$$L_0 = \begin{pmatrix} 0 & i \operatorname{curl} \\ -i \operatorname{curl} & 0 \end{pmatrix}, \qquad M = \begin{pmatrix} \frac{\varepsilon}{c} & 0 \\ 0 & \frac{\mu}{c} \end{pmatrix}, \qquad B = \begin{pmatrix} i\frac{4\pi\sigma}{c} & 0 \\ 0 & 0 \end{pmatrix}.$$

The operator L_0 is known as the Maxwell operator. System (26), (27) is equivalent to the equation,

$$(L_1 - \omega)\mathbf{u} = \mathbf{F}_1, \qquad \mathbf{u} = (\mathbf{E}, \mathbf{H})^T, \tag{29}$$

where

$$L_1 = L - M^{-1}B, \qquad L = M^{-1}L_0, \qquad \mathbf{F} = i4\pi c^{-1}(\mathbf{f}, 0)^T, \qquad \mathbf{F}_1 = M^{-1}\mathbf{F}.$$

It follows from (26) and (27) that

$$\operatorname{div}\bigl\{(\varepsilon\omega + i4\pi\sigma)\mathbf{E} + i4\pi\mathbf{f}\bigr\} = 0, \tag{30}$$

$$\operatorname{div}(\omega\mu\mathbf{H}) = 0 \quad \left(\text{or } -i \operatorname{div} \mathbf{H} = i\frac{\nabla\mu}{\mu}\mathbf{H}\right). \tag{31}$$

Clearly, system (26), (27), (30), (31) is overdetermined. We modify it, using an idea of "elliptization" of the time harmonic Maxwell equations [23,32] by introducing two scalar functions Φ and Ψ and passing to the system:

$$-i \operatorname{div} \mathbf{E} = i\frac{\nabla(\varepsilon\omega + i4\pi\sigma)}{\varepsilon\omega + i4\pi\sigma} \cdot \mathbf{E} - i\frac{c}{\varepsilon\omega + i4\pi\sigma}\Phi + i\frac{4\pi \operatorname{div}(i\mathbf{f})}{\varepsilon\omega + i4\pi\sigma}, \tag{32}$$

$$i \operatorname{curl} \mathbf{H} - i\nabla\Phi = \frac{\varepsilon\omega + i4\pi\sigma}{c}\mathbf{E} + i\frac{4\pi}{c}\mathbf{f}, \tag{33}$$

$$-i \operatorname{curl} \mathbf{E} - i\nabla\Psi = \frac{\mu\omega}{c}\mathbf{H}, \tag{34}$$

$$-i \operatorname{div} \mathbf{H} = i\frac{\nabla\mu}{\mu}\mathbf{H} - i\frac{c}{\mu\omega}\Psi. \tag{35}$$

We rewrite these equations in a matrix form. Let us introduce the matrices:

$$T = \begin{pmatrix} 0 & -i \operatorname{div} & 0 & 0 \\ -i\nabla & 0 & i \operatorname{curl} & 0 \\ 0 & -i \operatorname{curl} & 0 & -i\nabla \\ 0 & 0 & -i \operatorname{div} & 0 \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} \Phi \\ \mathbf{E} \\ \mathbf{H} \\ \Psi \end{pmatrix},$$

$$A_\omega = \begin{pmatrix} \frac{-ic}{\varepsilon\omega+i4\pi\sigma} & \frac{i\nabla(\varepsilon\omega+i4\pi\sigma)}{\varepsilon\omega+i4\pi\sigma} & 0 & 0 \\ 0 & \frac{\varepsilon\omega+i4\pi\sigma}{c} & 0 & 0 \\ 0 & 0 & \frac{\mu\omega}{c} & 0 \\ 0 & 0 & \frac{i\nabla\mu}{\mu} & \frac{-ic}{\mu\omega} \end{pmatrix}, \qquad \mathbf{F}_\omega = \begin{pmatrix} \frac{-4\pi\,\mathrm{div}\,\mathbf{f}}{\varepsilon\omega+i4\pi\sigma} \\ \frac{i4\pi\mathbf{f}}{c} \\ 0 \\ 0 \end{pmatrix}.$$

Then system (32)–(35) becomes,

$$T\mathbf{v} = A_\omega\mathbf{v} + \mathbf{F}_\omega. \tag{36}$$

To describe the domain $D(T)$ where the operator $T : L^2 \to L^2$ is defined, we set the following boundary conditions for $\mathbf{v}$ on $\partial\Omega$:

$$\mathbf{E}\wedge\mathbf{n} = 0, \qquad \mathbf{H}\cdot\mathbf{n} = 0, \qquad \Phi = 0, \qquad \nabla\Psi\cdot\mathbf{n} = 0. \tag{37}$$

Hence $D(T)$ can be described as

$$\begin{aligned} &\Phi \in H_0^1(\Omega), \qquad \mathbf{E} \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega), \\ &\mathbf{H} \in H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega), \qquad \Psi \in H^1(\Omega), \end{aligned} \tag{38}$$

where

$$H(\mathrm{div}, \Omega) = \{\mathbf{u} \in L^2(\Omega; \mathbb{C}^3),\ \mathrm{div}\,\mathbf{u} \in L^2(\Omega; \mathbb{C}^3)\},$$
$$H_0(\mathrm{div}, \Omega) = \{\mathbf{u} \in H(\mathrm{div}, \Omega),\ \mathbf{u}\cdot\mathbf{n} = 0\}.$$

Now, one can verify that the operator $T : D(T) \to L^2$ is self-adjoint and $D(T)$ is compactly immersed in L^2, which implies that the spectrum $\sigma(T)$ consists of isolated real points of finite multiplicity. Hence, there is a real k such that $(T-k)^{-1} \in \mathcal{L}(L^2, L^2)$ and $(T-k)^{-1}$ is compact. Eq. (36) is equivalent to

$$\mathbf{v} - (T-k)^{-1}(A_\omega - k)\mathbf{v} = (T-k)^{-1}\mathbf{F}_\omega. \tag{39}$$

The operator-valued function $\omega \to (T-k)^{-1}(A_\omega - k)$ is analytic in $G = \mathbb{C} \setminus \Gamma$, where

$$\Gamma = \{\omega \in \mathbb{C}\colon\ \omega = 0 \text{ or } \omega = -i4\pi\sigma/\varepsilon,\ \text{for some } x \in \Omega\} \subset i\mathbb{R}^-.$$

By the analytic Fredholm theorem [33], the operator $I - (T-k)^{-1}(A_\omega - k)$ is invertible for all $\omega \in G \setminus S$ where S is a discrete set in $\mathbb{C}$.

Lemma 1. *Let* $(\Phi, \mathbf{E}, \mathbf{H}, \Psi)^T \in D(T)$ *be a solution of problem* (36)–(38). *Then* $\Phi = \Psi = 0$ *and the vector* $(\mathbf{E}, \mathbf{H})^T$ *solves Eq.* (29)*, with*

$$(\mathbf{E}, \mathbf{H}) \in H_0(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega) \times H(\mathrm{curl}, \Omega) \cap H_0(\mathrm{div}, \Omega).$$

The inverse claim is also valid.

Proof. It follows from (32)–(35) and (37) that

$$\Delta\Phi = i\Phi, \qquad \Delta\Psi = i\Psi, \qquad \Phi|_{\partial\Omega} = 0, \qquad \nabla\Psi\cdot\mathbf{n}|_{\partial\Omega} = 0.$$

Hence $\Phi = \Psi = 0$. □

It follows from the energy equality (24) that problem (29) has a unique solution for any real ω, if $\sigma \geqslant c_0 > 0$. By the above equivalence, problem (39) has a zero solution, if $\mathbf{F}_\omega = 0$. By the Fredholm theorem, problem (39) is uniquely solvable for any $\mathbf{F}_\omega \in L^2$, if $\omega > 0$. We conclude that problem (3)–(5) is well-posed for $\omega > 0$, when ε, μ and σ are smooth functions.

Now, we study problem (3)–(5) for discontinuous coefficients. Given functions ε, μ and σ satisfying (25), we consider sequences of regular functions $\varepsilon_n, \mu_n, \sigma_n \in C^\infty(\Omega)$ such that

$$0 < c_0 \leqslant \varepsilon_n, \mu_n, \sigma_n \leqslant c_0^{-1}, \quad \text{uniformly in } n, \tag{40}$$

and

$$\varepsilon_n, \mu_n, \sigma_n \to \varepsilon, \mu, \sigma \quad \text{in } L^2(\Omega), \text{ as } n \to \infty, \tag{41}$$

respectively. By the above results, problem (13), with ε, μ, σ substituted by ε_n, μ_n, σ_n, has a unique solution $\mathbf{E}_n$ for all $\omega \in I$, where I is an interval:

$$I = \{\omega\colon \omega_1 < \omega < \omega_2\}, \quad 0 < \omega_1 < \omega_2.$$

It follows from the energy equalities (14) and (15) that the estimate,

$$\|\mathbf{E}_n\|_{H_0(\mathrm{curl},\Omega)} \leqslant b, \quad \forall \omega \in I,$$

holds uniformly in n. Then there is a function $\mathbf{E} \in H_0(\mathrm{curl}, \Omega)$ and a subsequence, still denoted $\mathbf{E}_n$, such that

$$\mathbf{E}_n \rightharpoonup \mathbf{E}, \qquad \mathrm{curl}\, \mathbf{E}_n \rightharpoonup \mathrm{curl}\, \mathbf{E} \quad \text{weakly in } L^2(\Omega).$$

According to (40) and (41), one can verify that $\mathbf{E}$ is the unique weak solution of problem (3)–(5) (corresponding to the limit functions ε, μ, and σ) for any $\omega \in I$.

We note that the analysis above can easily be adapted to problems (13), (16), (19) and (21). We have thus proved the following result:

Theorem 1. *Let $q \geqslant 0$, $\mathbf{f} \in H(\mathrm{div}, \Omega)$, and the conditions* (25) *be satisfied. Then, problems* (13), (16), (19) *and* (21) *are uniquely solvable for any $\omega > 0$.*

Note that the condition $\sigma \geqslant c_0 > 0$ is essential. Indeed, given $q \geqslant 0$, one can apply the Cauchy–Schwarz inequality to the right-hand sides of the energy equalities (15) and (18) to show that the solution $\mathbf{E}^\delta$ of problem (16) satisfies the estimate,

$$\int_\Omega \left(\left|\mathbf{E}^\delta\right|^2 + \delta^{2q}\left|\mathrm{curl}\, \mathbf{E}^\delta\right|^2\right) dx \leqslant b, \quad \forall \omega \in I, \tag{42}$$

where the constant b does not depend on δ. The solution of problem (13) satisfies the uniform estimate (42) with $q = 0$, while the solutions of problems (19) and (21) satisfy (42) with $q = 1$.

5. Homogenization

In this section we investigate the asymptotic behavior of the electromagnetic fields $\mathbf{E}^\delta$, $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$ as $\delta \to 0$. We assume that $\mathbf{f} \in H(\mathrm{div}, \Omega)$ and ε, μ and σ are given Y-periodic functions in $L^\infty(\mathbb{R}^3)$ such that

$$0 < c_0 \leqslant \varepsilon, \mu, \sigma \leqslant c_0^{-1}. \tag{43}$$

We consider successively the low frequency case, high and very high frequency cases, quasi-stationary and low conductivity cases, and the medium frequency case.

Our approach is based on the notion of two-scale (multi-scale) convergence [1,2,30] introduced by Nguetseng and then developed by Allaire. A sequence u_δ of functions in $L^2(\Omega)$ is said two-scale convergent to a function $u_0(x, y)$, $u_0 \in L^2(\Omega \times Y)$, as $\delta \to 0$, if

$$\lim_{\delta \to 0} \int_\Omega u_\delta(x)\varphi(x, x/\delta)\, dx = \frac{1}{|Y|} \int_\Omega \int_Y u_0(x, y)\varphi(x, y)\, dx\, dy, \quad \forall \varphi \in C\big(\Omega; \mathcal{C}^\infty_{per}(Y)\big);$$

we will write $u_\delta \xrightarrow{2s} u_0(x, y)$. We emphasize that for each $x \in \Omega$, the test function $\varphi(x, \cdot)$ is Y-periodic in the variable y and belongs to $\mathcal{C}^\infty(Y)$. It is a crucial property of the two-scale convergence that for any bounded sequence of $L^2(\Omega)$ there is a subsequence, still denoted u_δ, and a function $u_0(x, y)$, $u_0 \in L^2(\Omega \times Y)$, such that $u_\delta \xrightarrow{2s} u_0(x, y)$ [1,30].

We also use the div–curl lemma of Murat and Tartar which is the basic result of the compensated compactness theory [27,28,43,44]. In its classical form the lemma states that

$$\mathbf{u}_\delta \cdot \mathbf{v}_\delta \rightharpoonup \mathbf{u} \cdot \mathbf{v} \quad \text{in } \mathcal{D}'(\Omega), \text{ as } \delta \to 0,$$

provided the sequences of vector functions $\mathbf{u}_\delta$ and $\mathbf{v}_\delta$ satisfy the conditions,

$$\mathbf{u}_\delta \rightharpoonup \mathbf{u} \quad \text{weakly in } H(\text{curl}, \Omega),$$

and

$$\mathbf{v}_\delta \rightharpoonup \mathbf{v} \quad \text{weakly in } H(\text{div}, \Omega).$$

Below, for convenience, extracted subsequences are not relabeled; they are denoted like the original sequences.

5.1. Low frequency case

We consider problem (13). Recall that ε, μ and σ are given Y-periodic functions in $L^\infty(\mathbb{R}^3)$ satisfying (43). We define a constant (homogenized) matrix μ^h by:

$$\mu^h_{jk} = \frac{1}{|Y|} \int_Y \mu(y)\left(\delta^k_j + \frac{\partial}{\partial y_j} w^k_\mu\right) dy, \tag{44}$$

where $w^k_\mu(y)$, $k = 1, 2, 3$, denotes a scalar Y-periodic function which solves the cell problem,

$$-\frac{\partial}{\partial y_j}\left(\mu(y)\frac{\partial}{\partial y_j} w^k_\mu\right) = \frac{\partial}{\partial y_j}\left(\mu(y)\delta^k_j\right), \qquad \int_Y w^k_\mu \, dy = 0. \tag{45}$$

Clearly, problem (45) has a unique weak solution $w^k_\mu(y) \in H^1_{per}(Y)$. Here $H^1_{per}(Y)$ denotes the space of function in $H^1_{loc}(\mathbb{R}^3)$ which are Y-periodic. We also define the constant (homogenized) matrix κ^{2h} by:

$$\kappa^{2h}_{kj} = \alpha^2_w \omega^2 \varepsilon^h_{kj} + i4\pi\alpha_s \omega \sigma^h_{kj}, \tag{46}$$

where

$$\begin{aligned} \varepsilon^h_{kj} &= \frac{1}{|Y|} \int_Y \varepsilon(y)\left(\delta^k_j + \frac{\partial}{\partial y_j} w^k\right) dy, \\ \sigma^h_{kj} &= \frac{1}{|Y|} \int_Y \sigma(y)\left(\delta^k_j + \frac{\partial}{\partial y_j} w^k\right) dy, \end{aligned} \tag{47}$$

and $w^k(y) \in H^1_{per}(Y)$, $k = 1, 2, 3$, is the solution of the cell problem

$$-\frac{\partial}{\partial y_j}\left(\kappa^2(y)\frac{\partial}{\partial y_j} w^k\right) = \frac{\partial}{\partial y_j}\left(\kappa^2(y)\delta^k_j\right), \qquad \int_Y w^k \, dy = 0. \tag{48}$$

Observe that the homogenized magnetic permeability μ^h does not depend on the angular frequency ω, in contrast to ε^h and σ^h.

Let us establish the following result:

Theorem 2. *The sequence $\mathbf{E}^\delta$ of solutions of problem* (13) *converges weakly in $H(\text{curl}, \Omega)$, as $\delta \to 0$, to a solution $\mathbf{E}$ of the homogenized equation*:

$$\int_\Omega \left(\mu^h\right)^{-1} \text{curl}\,\mathbf{E} \cdot \text{curl}\,\mathbf{u}^* - \left(\kappa^2\right)^h \mathbf{E} \cdot \mathbf{u}^* - i4\pi\alpha^2_s \mathbf{f} \cdot \mathbf{u}^* \, dx = 0, \quad \forall \mathbf{u} \in H_0(\text{curl}, \Omega), \tag{49}$$

where μ^h and $(\kappa^2)^h$ are the homogenized matrices defined by (44), (45), *and* (46)–(48) *respectively. Moreover, the sequences $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$, defined by $\mathbf{D}^\delta = \varepsilon_\delta \mathbf{E}^\delta$ and $\mathbf{J}^\delta = \sigma_\delta \mathbf{E}^\delta$, converge weakly in $L^2(\Omega)$, as $\delta \to 0$, to the fields $\mathbf{D}$ and $\mathbf{J}$, respectively, defined by*:

$$\mathbf{D} = \varepsilon^h \mathbf{E}, \qquad \mathbf{J} = \sigma^h \mathbf{E}, \tag{50}$$

where the homogenized matrices ε^h *and* σ^h *are defined by* (47)*,* (48).

Proof. To prove (49), we apply the div–curl lemma as in [9]. For any constant vector $\boldsymbol{\eta} \in \mathbb{R}^3$, we denote:

$$\mathbf{U}_\eta = \eta_k \nabla_y w_\mu^k + \boldsymbol{\eta},$$

where $w_\mu^k(y) \in H_{per}^1(Y)$ is the solution of the cell problem (45), $k = 1, 2, 3$. With the matrix μ^h defined by (44), one can verify that

$$\mu^h \boldsymbol{\eta} = \frac{1}{|Y|} \int_Y \mu(y) \mathbf{U}_\eta(y)\, dy, \tag{51}$$

and the vector function $\mathbf{U}_\eta(y)$ satisfies the conditions,

$$\operatorname{div}_y(\mu \mathbf{U}_\eta) = 0, \qquad \operatorname{curl}_y \mathbf{U}_\eta = 0, \qquad \frac{1}{|Y|} \int_Y \mathbf{U}_\eta\, dy = \boldsymbol{\eta}.$$

According to estimate (42) with $q = 0$, one can extract subsequences such that

$$\mathbf{E}^\delta \xrightarrow{\text{in } H(\operatorname{curl}, \Omega)} \mathbf{E}, \qquad \mu_\delta^{-1} \operatorname{curl} \mathbf{E}^\delta \xrightarrow{\text{in } L^2(\Omega)} \boldsymbol{\xi}, \qquad \kappa_\delta^2 \mathbf{E}^\delta \xrightarrow{\text{in } L^2(\Omega)} \boldsymbol{\zeta},$$

where the symbol $\rightharpoonup$ stands for the weak convergence. Clearly,

$$\operatorname{curl} \boldsymbol{\xi} - \boldsymbol{\zeta} = i 4\pi \alpha_s^2 \omega \mathbf{f}.$$

Now, given $g \in L^2(\Omega)$, we introduce the scalar function $\varphi_\delta \in H_0^1(\Omega)$ which solves the problem:

$$-\operatorname{div}\big(\mu_\delta(x) \nabla \varphi_\delta\big) = g, \quad \varphi_\delta|_{\partial\Omega} = 0, \tag{52}$$

and set $\mathbf{p}_\delta = \mu_\delta \nabla \varphi_\delta$. According to the estimate,

$$c_0 \|\nabla \varphi_\delta\|_{L^2(\Omega)} \leqslant \|g\|_{L^2(\Omega)},$$

one can extract subsequences such that

$$\varphi_\delta \rightharpoonup \varphi \quad \text{weakly in } H_0^1(\Omega), \qquad \mathbf{p}_\delta \rightharpoonup \mathbf{p}_0 \quad \text{weakly in } L^2(\Omega).$$

We have:

$$\mathbf{p}_0(x) \cdot \boldsymbol{\eta} = \nabla \varphi(x) \cdot \big(\mu^h \boldsymbol{\eta}\big). \tag{53}$$

Indeed, denoting

$$\mathbf{q}_\delta(x) = \mu_\delta(x) \mathbf{U}_\eta(x/\delta),$$

we pass to the limit in the identity,

$$\mathbf{p}_\delta(x) \cdot \mathbf{U}_\eta(x/\delta) = \nabla \varphi_\delta(x) \cdot \mathbf{q}_\delta(x).$$

By periodicity [34] (see also (51)),

$$\mathbf{U}_\eta(\cdot/\delta) \rightharpoonup \boldsymbol{\eta}, \qquad \mathbf{q}_\delta \rightharpoonup \mu^h \boldsymbol{\eta} \quad \text{weakly in } L^2(\Omega).$$

We have:

$$\operatorname{div} \mathbf{p}_\delta = -g \in L^2(\Omega), \qquad \operatorname{curl} \mathbf{U}_\eta(x/\delta) = 0, \qquad \operatorname{curl} \nabla \varphi_\delta = 0, \qquad \operatorname{div} \mathbf{q}_\delta = 0,$$

then equality (53) follows by the div–curl lemma. Since the vector $\boldsymbol{\eta}$ is arbitrary, we obtain that

$$\mathbf{p}_\delta \rightharpoonup \big(\mu^h\big)^T \nabla \varphi \quad \text{weakly in } H(\operatorname{div}, \Omega)$$

where $(\mu^h)^T$ stands for the adjoint matrix. Since μ^h is a real and symmetric matrix we have:

$$\mathbf{p}_\delta \rightharpoonup \mu^h \nabla\varphi \quad \text{weakly in } H(\operatorname{div}, \Omega). \tag{54}$$

Let us pass to the limit in the identity,

$$\mathbf{p}_\delta(x) \cdot \mu_\delta^{-1} \operatorname{curl} \mathbf{E}^\delta = \operatorname{curl} \mathbf{E}^\delta \cdot \nabla\varphi_\delta.$$

In addition to the convergence (54), we have that $\operatorname{div} \operatorname{curl} \mathbf{E}_\delta = 0$, $\operatorname{curl} \nabla\varphi_\delta = 0$, and

$$\mu_\delta^{-1} \operatorname{curl} \mathbf{E}^\delta \rightharpoonup \boldsymbol{\xi} \quad \text{weakly in } H(\operatorname{curl}, \Omega).$$

Hence, by the div–curl lemma,

$$\left(\mu^h \boldsymbol{\xi} - \operatorname{curl} \mathbf{E}\right) \cdot \nabla\varphi = 0, \tag{55}$$

where the function $\varphi \in H_0^1(\Omega)$ solves the problem

$$-\operatorname{div}\left(\mu^h \nabla\varphi\right) = g, \quad \varphi|_{\partial\Omega} = 0.$$

Since the function g is arbitrary in $L^2(\Omega)$, the function φ spans a set which is dense in $H_0^1(\Omega)$, and equality (55) implies that

$$\mu_\delta^{-1} \operatorname{curl} \mathbf{E}^\delta \rightharpoonup \boldsymbol{\xi} = \left(\mu^h\right)^{-1} \operatorname{curl} \mathbf{E} \quad \text{weakly in } L^2(\Omega).$$

To study the limit of $\kappa_\delta^2 \mathbf{E}_\delta$, as $\delta \to 0$, we introduce the elliptic system,

$$\begin{aligned} &-\operatorname{div}\left(\alpha_w^2 \omega^2 \varepsilon_\delta(x) \nabla\varphi_{1\delta} - 4\pi\alpha_s^2 \omega\sigma_\delta(x) \nabla\varphi_{2\delta}\right) = g_1, \quad \varphi_{1\delta}|_{\partial\Omega} = 0, \\ &-\operatorname{div}\left(4\pi\alpha_s^2 \omega\sigma_\delta(x) \nabla\varphi_{1\delta} + \alpha_w^2 \omega^2 \varepsilon_\delta(x) \nabla\varphi_{2\delta}\right) = g_2, \quad \varphi_{2\delta}|_{\partial\Omega} = 0, \end{aligned} \tag{56}$$

for a vector-valued function $\boldsymbol{\varphi}_\delta = (\varphi_{1\delta}, \varphi_{2\delta})^T$, with $g_k \in L^2(\Omega)$, $k = 1, 2$. The bilinear form,

$$A(\boldsymbol{\varphi}, \boldsymbol{\Phi}) = \int_\Omega \left(\alpha_w^2 \omega^2 \varepsilon_\delta(x) \nabla\varphi_1 - 4\pi\alpha_s^2 \omega\sigma_\delta(x) \nabla\varphi_2\right) \cdot \nabla\Phi_1 + \left(4\pi\alpha_s^2 \omega\sigma_\delta(x) \nabla\varphi_1 + \alpha_w^2 \omega^2 \varepsilon_\delta(x) \nabla\varphi_2\right) \cdot \nabla\Phi_2 \, dx,$$

is coercive on $H_0^1(\Omega) \times H_0^1(\Omega)$ because of the equality

$$A(\boldsymbol{\varphi}, \boldsymbol{\varphi}) = \int_\Omega \alpha_w^2 \omega^2 \varepsilon_\delta(x) \left(|\nabla\varphi_1|^2 + |\nabla\varphi_2|^2\right) dx.$$

By the Lax–Milgram theorem, problem (56) has a unique solution in $H_0^1(\Omega) \times H_0^1(\Omega)$. In terms of the complex-valued functions $g = g_1 + ig_2$ and $\varphi_\delta = \varphi_{1\delta} + i\varphi_{2\delta}$, problem (56) becomes:

$$-\operatorname{div}\left(\kappa_\delta^2(x) \nabla\varphi_\delta\right) = g, \quad \varphi_\delta|_{\partial\Omega} = 0.$$

Let us pass to the limit in the identity:

$$\kappa_\delta^2 \mathbf{E}^\delta \cdot \nabla\varphi_\delta = \mathbf{E}^\delta \cdot \kappa_\delta^2 \nabla\varphi_\delta. \tag{57}$$

As in the case of Eq. (52), one can show that there are subsequences such that

$$\nabla\varphi_\delta \rightharpoonup \nabla\varphi \quad \text{weakly in } H(\operatorname{curl}, \Omega)$$

and

$$\kappa_\delta^2 \nabla\varphi_\delta \rightharpoonup \left(\left(\kappa^2\right)^h\right)^T \nabla\varphi \quad \text{weakly in } H(\operatorname{div}, \Omega).$$

It follows from Eq. (6) that

$$\kappa_\delta^2 \mathbf{E}^\delta \rightharpoonup \boldsymbol{\zeta} \quad \text{in } H(\operatorname{div}, \Omega) \text{ weakly}$$

provided $\operatorname{div} \mathbf{f} \in L^2(\Omega)$. Applying the div–curl lemma, we obtain from (57) that

$$\boldsymbol{\zeta} \cdot \nabla\varphi = \mathbf{E} \cdot \left(\left(\kappa^2\right)^h\right)^T \nabla\varphi.$$

Then we deduce that $\boldsymbol{\zeta} = (\kappa^2)^h\mathbf{E}$. Hence, the limit function $\mathbf{E}$ is a solution of the homogenized equation (49). Since the solution to (49) is unique, we conclude that the whole sequence $\mathbf{E}^\delta$ of solutions of problem (13) converges weakly in $H(\text{curl}, \Omega)$, as $\delta \to 0$, to the solution $\mathbf{E}$ of the homogenized equation (49).

Let us now study the limit of the induction field $\mathbf{D}^\delta$ and current field $\mathbf{J}^\delta$. Clearly, both $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$ are uniformly bounded in the L^2-norm and then, up to subsequences, $\mathbf{D}^\delta \rightharpoonup \mathbf{D}$, $\mathbf{J}^\delta \rightharpoonup \mathbf{J}$ weakly in $L^2(\Omega)$, as $\delta \to 0$. By definition of κ_δ^2, we have:

$$\kappa_\delta^2\mathbf{E}^\delta = \alpha_w^2\omega^2\mathbf{D}^\delta + i4\pi\alpha_s^2\omega\mathbf{J}^\delta.$$

We have proved above that

$$\alpha_\omega^2\omega^2\varepsilon^h\mathbf{E} + i4\pi\alpha_s^2\omega\sigma^h\mathbf{E} = \alpha_w^2\omega^2\mathbf{D} + i4\pi\alpha_s^2\omega\mathbf{J}$$

but this does not (directly) imply (50). To establish (50), one can apply the two-scale convergence method. It follows from estimate (42) with $q = 0$ that

$$\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \mathbf{E}_0(x, y), \qquad \text{curl}\,\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \boldsymbol{\chi}(x, y),$$

for some functions $\mathbf{E}_0$, $\boldsymbol{\chi} \in L^2(\Omega \times Y)$. Integrating by parts we have, for any $\boldsymbol{\varphi} \in C(\Omega; \mathcal{C}^\infty_{per}(Y))$,

$$\int_\Omega \delta\,\text{curl}\,\mathbf{E}^\delta(x) \cdot \boldsymbol{\varphi}(x, x/\delta)\,dx = \int_\Omega \mathbf{E}^\delta(x) \cdot \left\{\delta\,\text{curl}_x\,\boldsymbol{\varphi}(x, y) + \text{curl}_y\,\boldsymbol{\varphi}(x, y)\right\}\Big|_{y=x/\delta}\,dx.$$

Passing to the limit, as $\delta \to 0$, we obtain:

$$\iint_{\Omega\times Y} \mathbf{E}_0\,\text{curl}_y\,\boldsymbol{\varphi}\,dx\,dy = 0,$$

from which it follows that $\text{curl}_y\,\mathbf{E}_0 = 0$. Arguing as in [34], we conclude that there exist a vector function $\mathbf{E}^1(x)$, $\mathbf{E}^1 \in L^2(\Omega)$, and scalar functions $v^k(y)$, $v^k \in H^1_{per}(Y)$, $k = 1, 2, 3$, such that

$$\mathbf{E}_0(x, y) = \mathbf{E}^1(x) + E^1_k(x)\nabla_y v^k(y), \quad \mathbf{E}^1(x) = \frac{1}{|Y|}\int_Y \mathbf{E}_0(x, y)\,dy. \tag{58}$$

One more property of the two-scale convergence [1,30] is that $\mathbf{E}^\delta \rightharpoonup \mathbf{E}^1$ weakly in $L^2(\Omega)$, as $\delta \to 0$. Hence $\mathbf{E}^1 = \mathbf{E}$ where $\mathbf{E}$ is the solution of (49).

Let us now write Eq. (30) in the dimensionless variables:

$$\text{div}\big(\kappa_\delta^2\mathbf{E}^\delta + i4\pi\alpha_s^2\omega\mathbf{f}\big) = 0.$$

Given a scalar test function $\varphi(x, y)$, $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty_{per}(Y))$, we write the identity:

$$\begin{aligned}\int_\Omega \delta\,\text{div}\big(i4\pi\alpha_s^2\omega\mathbf{f}\big)\varphi\left(x, \frac{x}{\delta}\right)dx &= \int_\Omega \delta\kappa_\delta^2\mathbf{E}^\delta(x) \cdot \nabla\varphi\left(x, \frac{x}{\delta}\right)dx \\ &= \int_\Omega \kappa_\delta^2\mathbf{E}^\delta(x)\big\{\delta\nabla_x\varphi(x, y) + \nabla_y\varphi(x, y)\big\}\Big|_{y=x/\delta}\,dx.\end{aligned}$$

Passing to the limit, as $\delta \to 0$, we obtain:

$$\iint_{\Omega\times Y} \kappa^2(y)\mathbf{E}_0 \cdot \nabla_y\varphi\,dx\,dy = 0,$$

from which it follows that

$$\text{div}_y\big(\kappa^2(y)\mathbf{E}_0\big) = 0. \tag{59}$$

With $\mathbf{E}_0$ given by the representation formula (58), we deduce from (59) that the periodic scalar function $v^k(y)$ solves the boundary-value problem:

$$\operatorname{div}_y\big(\kappa^2(y)\big(\mathbf{e}^k+\nabla_y v^k\big)\big)=0,\qquad \frac{1}{|Y|}\int\limits_Y v^k\,dy=0,\quad k=1,2,3.$$

By uniqueness, $v^k=w^k$ where the function w^k solves problem (48).

Passing to the two-scale limit in the equality $\mathbf{D}^\delta=\varepsilon_\delta\mathbf{E}^\delta$, we have:

$$\mathbf{D}^\delta \stackrel{2s}{\rightharpoonup} \mathbf{D}_0(x,y),\quad \mathbf{D}_0(x,y)=\varepsilon(y)\mathbf{E}_0(x,y).$$

Denoting $\mathbf{D}(x)=\int_Y \mathbf{D}_0(x,y)\,dy/|Y|$ and making use of the representation formula (58), we integrate the equality $\mathbf{D}_0=\varepsilon(y)\mathbf{E}_0$ over the variable y to arrive at the first equation in (50). The second equation in (50) can be obtained similarly by passing to the two-scale limit in the equality $\mathbf{J}^\delta=\sigma_\delta\mathbf{E}^\delta$. The proof of Theorem 2 is complete. □

Remark 2. The proof of the first part of Theorem 2 related to Eq. (49) is based on the div–curl lemma and the oscillating test function method of Tartar. The same line of arguments can be applied for sure in deriving Eqs. (50), and yet we have switched to the two-scale convergence method.

Remark 3. In the particular case when $\omega=0$, the Maxwell equations (1) and (2) reduce to the equation,

$$\operatorname{curl}(A\operatorname{curl}\mathbf{H})=\operatorname{curl}\big(\sigma^{-1}\mathbf{f}\big),\tag{60}$$

with the matrix A given by $A_{ij}=c\delta_{ij}/(4\pi\sigma_\delta(x))$. Under the assumption $\operatorname{div}\mathbf{H}=0$, the two-scale convergence for Eq. (60) was studied in [47] when the matrix A is nonlinear, i.e. $A_{ij}=A_{ij}(\mathbf{H},x,x/\delta)$. Observe that, due to the equation $\operatorname{div}\mathbf{H}=0$, the sequence $\mathbf{H}^\delta$ is compact weakly not only in $H(\mathrm{curl},\Omega)$ but in $W_2^1(\Omega)$ also. One more observation is that in the case when the matrix A does not depend on $\mathbf{H}$ Eq. (60) can be associated with a coercive operator.

5.2. High and very high frequency cases

Here we study the homogenization of Eq. (8) for $q\geqslant 1$. We distinguish two cases: $q=1$ and $q>1$:

5.2.1. Case $q=1$

For $q=1$, Eq. (8) becomes,

$$\delta^2\operatorname{curl}\big(\mu_\delta^{-1}\operatorname{curl}\mathbf{E}^\delta\big)-k_{1\delta}^2\mathbf{E}^\delta=i4\pi\omega\alpha_s^2\mathbf{f},$$

with $k_{1\delta}^2(x)=\alpha_w^2\omega^2\varepsilon_\delta(x)+i4\pi\alpha_s^2\omega\sigma_\delta(x)$. Let $H_{per}(\mathrm{curl},Y)$ denote the space of functions in $H_{loc}(\mathrm{curl},\mathbb{R}^3)$ which are Y-periodic. We introduce a vector-valued function $\mathbf{M}^j(y)\in H_{per}(\mathrm{curl},Y)$, $j=1,2,3$, which solves the cell problem,

$$\operatorname{curl}_y\big(\mu^{-1}(y)\operatorname{curl}_y\mathbf{M}^j(y)\big)-\kappa_1^2(y)\mathbf{M}^j(y)=i4\pi\omega\alpha_s^2\mathbf{e}^j,\tag{61}$$

where $\{\mathbf{e}^1,\mathbf{e}^2,\mathbf{e}^3\}$ is the canonical basis of $\mathbb{R}^3$ and we denote by $M(y)$ the matrix whose column vectors are $\mathbf{M}^j$ i.e. $\mathbf{M}^j(y)=M(y)\,\mathbf{e}^j$.

Lemma 2. *For any $j=1,2,3$, problem* (61) *has a unique weak solution such that $\mathbf{M}^j\in H_{per}(\mathrm{curl},Y)$, and*

$$\int\limits_Y \mu^{-1}\operatorname{curl}\mathbf{M}^j\cdot\operatorname{curl}\mathbf{u}^*-\kappa_1^2\mathbf{M}^j\cdot\mathbf{u}^*\,dy=i4\pi\omega\alpha_s^2\int\limits_Y \mathbf{e}^j\cdot\mathbf{u}^*\,dy,\quad \forall\mathbf{u}\in H_{per}(\mathrm{curl},Y).$$

Proof. The proof is analogous to that of Theorem 1, so we only indicate the main modifications. One can verify that the function $\mathbf{M}^j$ is a solution of (61) if and only if the pair $(\mathbf{M}^j,\mathbf{N}^j)\in H_{per}(\mathrm{curl},Y)\times H_{per}(\mathrm{curl},Y)$ is a solution of the system:

$$\begin{aligned}&i\omega\alpha_\omega\mu\mathbf{N}^j-\operatorname{curl}\mathbf{M}^j=0,\\ &i\omega\alpha_\omega\operatorname{curl}\mathbf{N}^j-\kappa_1^2\mathbf{M}^j-i4\pi\omega\alpha_s^2\mathbf{e}^j=0.\end{aligned}$$

The idea of elliptization consists in introducing two scalar functions Φ^j, Ψ^j and passing to the extended system:

$$-i\,\mathrm{div}\,\mathbf{M}^j = i\frac{\nabla k_1^2}{k_1^2}\cdot\mathbf{M}^j - i\frac{\omega\alpha_\omega}{k_1^2}\Phi^j,$$
$$i\,\mathrm{curl}\,\mathbf{N}^j - i\nabla\Phi^j = \frac{k_1^2}{\omega\alpha_\omega}\mathbf{M}^j + \frac{i4\pi\alpha_s^2}{\alpha_\omega}\mathbf{e}^j,$$
$$-i\,\mathrm{curl}\,\mathbf{M}^j - i\nabla\Psi^j = \omega\alpha_\omega\mu\mathbf{N}^j,$$
$$-i\,\mathrm{div}\,\mathbf{N}^j = i\frac{\nabla\mu}{\mu}\cdot\mathbf{N}^j - i\frac{1}{\omega\alpha_\omega\mu}\Psi^j,$$

which can be written in a matrix form. Denote:

$$T = \begin{pmatrix} 0 & -i\,\mathrm{div} & 0 & 0 \\ -i\nabla & 0 & i\,\mathrm{curl} & 0 \\ 0 & -i\,\mathrm{curl} & 0 & -i\nabla \\ 0 & 0 & -i\,\mathrm{div} & 0 \end{pmatrix}, \qquad \mathbf{v}^j = \begin{pmatrix} \Phi^j \\ \mathbf{M}^j \\ \mathbf{N}^j \\ \Psi^j \end{pmatrix},$$

$$\widetilde{A}_\omega = \begin{pmatrix} \frac{-i\omega\alpha_\omega}{k_1^2} & \frac{i\nabla k_1^2}{k_1^2} & 0 & 0 \\ 0 & \frac{k_1^2}{\omega\alpha_\omega} & 0 & 0 \\ 0 & 0 & \omega\alpha_\omega\mu & 0 \\ 0 & 0 & \frac{i\nabla\mu}{\mu} & \frac{-i}{\omega\alpha_\omega\mu} \end{pmatrix}, \qquad \widetilde{\mathbf{F}}_\omega = \begin{pmatrix} 0 \\ \frac{i4\pi\alpha_s^2\mathbf{e}^j}{\alpha_\omega} \\ 0 \\ 0 \end{pmatrix}.$$

The system becomes,

$$\widetilde{T}\mathbf{v}^j = \widetilde{A}_\omega\mathbf{v}^j + \widetilde{\mathbf{F}}_\omega.$$

The domain $D(\widetilde{T})$ where the operator $\widetilde{T} : L^2 \to L^2$ is defined can be described as follows:

$$\Phi^j \in H^1_{per}(Y), \qquad \mathbf{M}^j \in H_{per}(\mathrm{curl}, Y) \cap H_{per}(\mathrm{div}, Y),$$
$$\mathbf{N}^j \in H_{per}(\mathrm{curl}, Y) \cap H_{per}(\mathrm{div}, Y), \qquad \Psi^j \in H^1_{per}(Y),$$

where $H_{per}(\mathrm{div}, Y)$ denotes the space of functions in $H_{loc}(\mathrm{div}, Y)$ which are Y-periodic.

Then, by arguments similar to those presented in Section 3 we show that problem (61) has a unique weak solution. □

We formulate the main result of this section.

Theorem 3. *The sequence $\mathbf{E}^\delta$ of solutions of problem* (16)*, with $q = 1$, converges in $L^2(\Omega)$ weakly, as $\delta \to 0$, to the field $\mathbf{E}$ defined by*:

$$\mathbf{E} = M^h\mathbf{f}, \quad M^h = \frac{1}{|Y|}\int_Y M(y)\,dy, \tag{62}$$

where $M(y)$ is the Y-periodic matrix whose column vectors $\mathbf{M}^j$ ($j = 1, 2, 3$) are given by the cell problem (61)*. Moreover, the sequences $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$, defined by $\mathbf{D}^\delta = \varepsilon_\delta\mathbf{E}^\delta$ and $\mathbf{J}^\delta = \sigma_\delta\mathbf{E}^\delta$, converge in $L^2(\Omega)$ weakly, as $\delta \to 0$, to the fields $\mathbf{D}$ and $\mathbf{J}$, respectively, defined by*:

$$\mathbf{D} = \varepsilon^h\mathbf{E}, \qquad \mathbf{J} = \sigma^h\mathbf{E},$$

where the homogenized matrices ε^h and σ^h are defined by,

$$\varepsilon^h = \frac{1}{|Y|}\int_Y \varepsilon(y)M(y)\,dy\,\left(M^h\right)^{-1}, \qquad \sigma^h = \frac{1}{|Y|}\int_Y \sigma(y)M(y)\,dy\,\left(M^h\right)^{-1}. \tag{63}$$

Proof. It follows from estimate (42) that there are subsequences such that

$$\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \mathbf{E}_0(x,y), \qquad \delta \operatorname{curl} \mathbf{E}^\delta \overset{2s}{\rightharpoonup} \boldsymbol{\chi}(x,y), \tag{64}$$

for some functions $\mathbf{E}_0,\ \boldsymbol{\chi} \in L^2(\Omega \times Y)$. Moreover,

$$\boldsymbol{\chi}(x,y) = \operatorname{curl}_y \mathbf{E}_0(x,y). \tag{65}$$

Indeed, integrating by parts we have, for any $\boldsymbol{\varphi} \in C(\Omega; \mathcal{C}^\infty_{per}(Y))$,

$$\int_\Omega \delta \operatorname{curl} \mathbf{E}^\delta(x) \cdot \boldsymbol{\varphi}(x, x/\delta)\, dx = \int_\Omega \mathbf{E}^\delta(x) \cdot \big\{\delta \operatorname{curl}_x \boldsymbol{\varphi}(x,y) + \operatorname{curl}_y \boldsymbol{\varphi}(x,y)\big\}\big|_{y=x/\delta}\, dx,$$

hence

$$\int_\Omega\int_Y \boldsymbol{\chi} \cdot \boldsymbol{\varphi}(x,y)\, dx\, dy = \int_\Omega\int_Y \mathbf{E}_0(x,y) \cdot \operatorname{curl}_y \boldsymbol{\varphi}(x,y)\, dx\, dy$$

and equality (65) follows.

Taking $\mathbf{u}^*(x) = \boldsymbol{\varphi}(x, x/\delta)$ as a test function in (16), with $\boldsymbol{\varphi} \in L^2(\Omega; \mathcal{C}^\infty_{per}(Y))$, and applying regularization of the functions $\mu(y)$ and $\kappa_1^2(y)$, we arrive, as $\delta \to 0$, at the limit equation,

$$\operatorname{curl}_y\big(\mu^{-1}(y) \operatorname{curl}_y \mathbf{E}_0(x,y)\big) - \kappa_1^2(y)\mathbf{E}_0(x,y) = i4\pi\alpha_s^2\omega\mathbf{f}(x). \tag{66}$$

To derive a macroscopic equation for the mean field,

$$\mathbf{E}(x) = \widetilde{\mathbf{E}_0(x,y)} \equiv \frac{1}{|Y|}\int_Y \mathbf{E}_0(x,y)\, dy,$$

which coincides with the weak limit of $\mathbf{E}^\delta$ in $L^2(\Omega)$, we look for a solution $\mathbf{E}_0(x,y)$ of Eq. (66) in the form,

$$\mathbf{E}_0(x,y) = N(y)\mathbf{f}(x), \quad \text{i.e. } E_{0j} = N_{jk}(y) f_k(x),$$

with an unknown Y-periodic matrix $N(y)$. Putting such a field $\mathbf{E}_0(x,y)$ into (66) we obtain that the column vector N^j $(j = 1,2,3)$ of $N(y)$ is a Y-periodic solution of the cell problem (61), and $N = M$ by uniqueness. Consequently, the limit field $\mathbf{E}$ satisfies Eq. (62).

Consider now the induction field $\mathbf{D}^\delta = \varepsilon_\delta \mathbf{E}^\delta$. Clearly,

$$\mathbf{D}^\delta \overset{2s}{\rightharpoonup} \mathbf{D}_0(x,y) = \varepsilon(y)\mathbf{E}_0(x,y),$$

then $\mathbf{D}^\delta \rightharpoonup \mathbf{D}$ weakly in $L^2(\Omega)$ with

$$\begin{aligned}\mathbf{D}(x) = \widetilde{\mathbf{D}_0(x,y)} &= \frac{1}{|Y|}\int_Y \varepsilon(y)\mathbf{E}_0(x,y)\, dy = \frac{1}{|Y|}\int_Y \varepsilon(y)M(y)\,\mathbf{f}(x)\, dy \\ &= \frac{1}{|Y|}\int_Y \varepsilon(y)M(y)\big(M^h\big)^{-1}\mathbf{E}(x)\, dy = \varepsilon^h\mathbf{E}(x),\end{aligned}$$

where ε^h is defined by (63).

Similarly, we show that there is a subsequence of $\mathbf{J}^\delta = \sigma_\delta\mathbf{E}^\delta$ which converges weakly in $L^2(\Omega)$, as $\delta \to 0$, to the field $\mathbf{J}$ given by $\mathbf{J} = \sigma^h\mathbf{E}$ where the homogenized matrix σ^h is given by (63). Theorem 3 is proved. □

Remark 4. Denoting

$$\big(\kappa_1^2\big)^h = \frac{1}{|Y|}\int_Y \kappa_1^2(y)M(y)\, dy\, \big(M^h\big)^{-1},$$

we observe that $(\kappa_1^2)^h = -i4\pi\alpha_s^2\omega(M^h)^{-1}$, then it is easy to see that the limit field $\mathbf{E}$ satisfies the homogenized equation,

$$-\left(\kappa_1^2\right)^h \mathbf{E} = i4\pi\alpha_s^2\omega\mathbf{f}.$$

Remark 5. Under hypothesis (7) with $q = 1$, the second equation of system (1) for the magnetic field $\mathbf{H}^\delta$ in the dimensionless variables becomes:

$$\mathbf{H}^\delta = -\frac{i\delta \operatorname{curl} \mathbf{E}^\delta}{\mu_\delta(x)\omega\alpha_H}, \quad \alpha_H = \frac{\hat{E}\sqrt{\hat{\varepsilon}/\hat{\mu}}}{\hat{H}\alpha_w}.$$

It follows from estimate (42) and the convergence (64) that

$$\mathbf{H}^\delta \overset{2s}{\rightharpoonup} \mathbf{H}_0(x, y) = -\frac{i \operatorname{curl}_y \mathbf{E}_0(x, y)}{\mu(y)\omega\alpha_H}.$$

Then $\mathbf{H}^\delta \rightharpoonup \mathbf{H}$ weakly in $L^2(\Omega)$ and the limit fields $\mathbf{E}$ and $\mathbf{H}$ are related by:

$$\mathbf{H} = -\frac{i}{\omega\alpha_H} N_\mu^h \left(M^h\right)^{-1}\mathbf{E}, \quad N_\mu^h = \frac{1}{|Y|}\int_Y \frac{\operatorname{curl} M(y)}{\mu(y)}\, dy.$$

Thus, as far as the homogenized magnetic permeability is concerned, it is unlikely that the equality,

$$\mathbf{H} = -\frac{i}{\omega\alpha_H}\left(\mu^h\right)^{-1} \operatorname{curl}\mathbf{E},$$

holds for some constant matrix μ^h.

5.2.2. Case $q > 1$

Here we consider Eq. (8) which can be written as

$$\delta^{2q} \operatorname{curl}\left(\mu_\delta^{-1} \operatorname{curl}\mathbf{E}^\delta\right) - k_{1\delta}^2\mathbf{E}^\delta = i4\pi\omega\alpha_s^2\mathbf{f},$$

with $\kappa_{1\delta}^2 = \alpha_w^2\omega^2\varepsilon_\delta + i4\pi\alpha_s^2\omega\sigma_\delta$.

We will prove the following result:

Theorem 4. *The sequence $\mathbf{E}^\delta$ of solutions of problem* (16)*, with $q > 1$, converges weakly in $L^2(\Omega)$, as $\delta \to 0$, to the field $\mathbf{E}$ given by*:

$$\mathbf{E} = m^h\mathbf{f}, \quad m^h = -\frac{i4\pi\omega\alpha_s^2}{|Y|}\int_Y \frac{1}{\kappa_1^2(y)}\, dy. \tag{67}$$

Moreover, the sequences $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$, defined by $\mathbf{D}^\delta = \varepsilon_\delta\mathbf{E}^\delta$ and $\mathbf{J}^\delta = \sigma_\delta\mathbf{E}^\delta$, converge weakly in $L^2(\Omega)$, as $\delta \to 0$, to the fields $\mathbf{D}$ and $\mathbf{J}$, respectively, defined by:

$$\mathbf{D} = \varepsilon^h\mathbf{E}, \qquad \mathbf{J} = \sigma^h\mathbf{E},$$

where the homogenized constants ε^h and σ^h are defined by,

$$\varepsilon^h = \frac{1}{m^h|Y|}\int_Y \varepsilon(y)m(y)\, dy, \qquad \sigma^h = \frac{1}{m^h|Y|}\int_Y \sigma(y)m(y)\, dy.$$

Proof. According to estimate (42) we have, up to a subsequence, that

$$\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \mathbf{E}_0(x, y), \quad \mathbf{E}_0 \in L^2(\Omega \times Y).$$

Taking $\mathbf{u}^*(x) = \boldsymbol{\varphi}(x, x/\delta)$ as a test function in (16), with $\boldsymbol{\varphi} \in L^2(\Omega; \mathcal{C}_{per}^\infty(Y))$, we obtain:

$$\int_{\Omega} \delta^{2q}\mu_\delta^{-1} \operatorname{curl}\mathbf{E}\cdot \operatorname{curl}\boldsymbol{\varphi}(x,x/\delta)\,dx$$
$$= \int_{\Omega} \delta^{2q}\mu_\delta^{-1} \operatorname{curl}\mathbf{E}\cdot \big(\operatorname{curl}_x \boldsymbol{\varphi}(x,y)+\delta^{-1}\operatorname{curl}_y \boldsymbol{\varphi}(x,y)\big)\big|_{y=x/\delta}\,dx.$$

Since $\delta^q \operatorname{curl}\mathbf{E}^\delta$ is bounded in $L^2(\Omega)$ and $q>1$, we deduce that

$$\lim_{\delta\to 0}\int_{\Omega} \delta^{2q}\mu_\delta^{-1} \operatorname{curl}\mathbf{E}\cdot \operatorname{curl}\boldsymbol{\varphi}(x,x/\delta)\,dx = 0,$$

then equality (16) with $\mathbf{u}^*(x)=\boldsymbol{\varphi}(x,x/\delta)$ gives:

$$-\frac{1}{|Y|}\int_{\Omega}\int_{Y} \kappa_1^2(y)\mathbf{E}_0(x,y)\varphi(x,y)\,dx\,dy = \frac{i4\pi\omega\alpha_s^2}{|Y|}\int_{\Omega}\int_{Y}\mathbf{f}(x)\varphi(x,y)\,dx\,dy.$$

We then have:

$$-\kappa_1^2(y)\mathbf{E}_0(x,y) = i4\pi\omega\alpha_s^2\mathbf{f}(x).$$

Taking $\mathbf{E}_0$ in the form $\mathbf{E}_0(x,y)=m(y)\mathbf{f}(x)$ we conclude that

$$m(y) = -\frac{i4\pi\omega\alpha_s^2}{\kappa_1^2(y)},$$

then the macroscopic field $\mathbf{E}(x)=\widetilde{\mathbf{E}_0(x,y)}$ satisfies (67). Arguing as in the proof of Theorem 3, we obtain the second part of Theorem 4. □

5.3. Quasi-stationary and low conductivity cases

The quasi-stationary case (9) is slightly different from the high frequency case (8) with $q=1$. So, we formulate only the final results: for subsequences not relabeled for convenience,

$$\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \mathbf{E}_0(x,y), \qquad \delta \operatorname{curl}\mathbf{E}^\delta \overset{2s}{\rightharpoonup} \operatorname{curl}_y \mathbf{E}_0(x,y), \tag{68}$$

with $\mathbf{E}_0, \operatorname{curl}_y\mathbf{E}_0 \in L^2(\Omega\times Y)$. Moreover, $\mathbf{E}_0(x,y)$ satisfies the equation

$$\operatorname{curl}_y\big(\mu^{-1}(y)\operatorname{curl}_y\mathbf{E}_0(x,y)\big) - i\omega 4\pi\sigma(y)\alpha_s^2\mathbf{E}_0(x,y) = i4\pi\alpha_s^2\omega\mathbf{f}(x).$$

The macroscopic field $\mathbf{E}(x)=\widetilde{\mathbf{E}_0(x,y)}$ is given by:

$$\mathbf{E} = M^h\mathbf{f}, \quad M^h = \widetilde{M(y)},$$

where $M(y)$ is the Y-periodic matrix whose column vectors $\mathbf{M}^j$ $(j=1,2,3)$ are given by the cell problem (61), with $\kappa_1^2(y)$ substituted by $\kappa_2^2(y)=i\omega 4\pi\alpha_s^2\sigma(y)$. With the matrix $M(y)$ at hand, the homogenized matrices ε^h and σ^h are given by the representation formulae (63). We also show that the sequences $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$ converge in $L^2(\Omega)$ weakly, as $\delta\to 0$, to $\mathbf{D}$ and $\mathbf{J}$, respectively, satisfying:

$$\mathbf{D} = \varepsilon^h\mathbf{E}, \qquad \mathbf{J} = \sigma^h\mathbf{E}.$$

The low conductivity case (10) is very degenerate. Indeed, in this case the convergences (68) hold, where the limit field $\mathbf{E}_0(x,y)$ solves the equation:

$$\operatorname{curl}_y\big(\mu^{-1}(y)\operatorname{curl}_y\mathbf{E}_0(x,y)\big) - \omega^2\varepsilon(y)\alpha_w^2\mathbf{E}_0(x,y) = 0.$$

Assume that both the coefficients $\mu(y)$ and $\varepsilon(y)$ are smooth then, by periodicity, it follows from (69) that $\int_Y \varepsilon(y)\mathbf{E}_0(x,y)\,dy = 0$. By passing to the two-scale limit in the equality $\mathbf{D}^\delta = \varepsilon_\delta\mathbf{E}^\delta$ we obtain $\mathbf{D}_0(x,y)=\varepsilon(y)\mathbf{E}_0(x,y)$. Hence, the weak limit $\mathbf{D}(x)=\widetilde{\mathbf{D}_0(x,y)}$ in $L^2(\Omega)$ of $\mathbf{D}^\delta$ is equal to zero.

5.4. Medium frequency case

The medium frequency case (7) can be considered as a problem with two length scales, δ^q and δ, the smallest length scale being δ. We use the multi-scale convergence method introduced in [2] as a generalization of the notion of two-scale convergence to the case of multiple separated scales of periodic oscillations. Let Z be a replica of Y:

$$Z = \left\{ z = (z_1, z_2, z_3):\ z \in \prod_{j=1}^{3} (0, r_j) \right\}.$$

The sequence u_δ of functions in $L^2(\Omega)$ is said 3-scale convergent to a function $u_0(x, z, y)$, $u_0 \in L^2(\Omega \times Z \times Y)$, as $\delta \to 0$, if

$$\lim_{\delta\to 0} \int_\Omega u_\delta(x)\varphi\big(x, x/\delta^q, x/\delta\big)\, dx = \frac{1}{|Y||Z|} \iiint_{\Omega\times Z\times Y} u_0(x, z, y)\varphi(x, z, y)\, dx\, dz\, dy,$$

for any $\varphi \in C(\overline{\Omega}; \mathcal{C}^\infty_{per}(Z \times Y))$. We will write the above convergence as $u_\delta \overset{3s}{\rightharpoonup} u_0(x, z, y)$ for short. We have the following compactness property [2]: for each bounded sequence u_δ in $L^2(\Omega)$ there is a subsequence, still denoted u_δ, and a function $u_0(x, z, y)$, $u_0 \in L^2(\Omega \times Z \times Y)$, such that $u_\delta \overset{3s}{\rightharpoonup} u_0(x, z, y)$.

We define the constant (homogenized) matrix $(\kappa_1^2)^h$ by:

$$\big(\kappa_1^2\big)^h_{ik} = \frac{1}{|Y|} \int_Y \kappa_1^2(y)\big(\delta_{ik} + \partial v^k/\partial y_i\big)\, dy, \tag{69}$$

where $v^k(y)$, $k = 1, 2, 3$, denotes the Y-periodic scalar function which solves the cell problem,

$$\operatorname{div}_y\big\{\kappa_1^2(y)\big(\mathbf{e}^k + \nabla_y v^k\big)\big\} = 0, \qquad \int_Y v^k\, dy = 0. \tag{70}$$

We also define the constant (homogenized) matrices ε^h and σ^h by:

$$\varepsilon^h_{ik} = \frac{1}{|Y|} \int_Y \varepsilon(y)\left(\delta_{ik} + \frac{\partial v^k(y)}{\partial y_i}\right) dy,$$
$$\sigma^h_{ik} = \frac{1}{|Y|} \int_Y \sigma(y)\left(\delta_{ik} + \frac{\partial v^k(y)}{\partial y_i}\right) dy. \tag{71}$$

We will prove the following result:

Theorem 5. *The sequence $\mathbf{E}^\delta$ of solutions of problem* (16)*, with $0 < q < 1$, converges weakly in $L^2(\Omega)$, as $\delta \to 0$, to the field $\mathbf{E}$ defined by the macroscopic equation,*

$$-\big(\kappa_1^2\big)^h \mathbf{E}(x) = i4\pi\alpha_s^2\omega\mathbf{f}(x), \tag{72}$$

where the constant matrix $(\kappa_1^2)^h$ is determined by (69) *via the solution of the cell problem* (70)*. Moreover, the sequences $\mathbf{D}^\delta$ and $\mathbf{J}^\delta$ converge weakly in $L^2(\Omega)$, as $\delta \to 0$, to the fields $\mathbf{D}$ and $\mathbf{J}$, respectively, defined by*:

$$\mathbf{D}(x) = \varepsilon^h\mathbf{E}(x), \qquad \mathbf{J}(x) = \sigma^h\mathbf{E}(x), \tag{73}$$

where the matrices ε^h and σ^h are defined by (71).

Proof. Estimate (42) implies that the sequences $\mathbf{E}^\delta$ and $\delta^q \operatorname{curl} \mathbf{E}^\delta$ are bounded in $L^2(\Omega)$, then there are subsequences which 3-scale converge:

$$\mathbf{E}^\delta \overset{3s}{\rightharpoonup} \mathbf{E}_0(x, z, y), \qquad \delta^q \operatorname{curl} \mathbf{E}^\delta \overset{3s}{\rightharpoonup} \boldsymbol{\chi}(x, z, y),$$

where $\mathbf{E}_0, \boldsymbol{\chi} \in L^2(\Omega \times Z \times Y)$. Note also that $\delta\mathbf{E}^\delta \to 0$ in $L^2(\Omega)$ strongly.

Consider now Eq. (30) which, in the dimensionless variables, becomes:

$$\operatorname{div}\left(\kappa_{1\delta}^2 \mathbf{E}^\delta + i4\pi\alpha_s^2\omega\mathbf{f}\right) = 0.$$

Given test functions $\varphi(x,z,y)$, $\varphi \in \mathcal{D}(\Omega; \mathcal{C}_{per}^\infty(Z \times Y))$, and $\varphi_1(x,z,y)$, $\varphi_1 \in \mathcal{D}(\Omega; \mathcal{C}_{per}^\infty(Z \times Y))$, $\operatorname{curl}_y \varphi_1 = 0$, we write the identities,

$$\begin{aligned}
&\int_\Omega \delta \operatorname{div}\left(i4\pi\alpha_s^2\omega\mathbf{f}\right)\varphi\left(x, \frac{x}{\delta^q}, \frac{x}{\delta}\right) dx = \int_\Omega \delta\kappa_{1\delta}^2 \mathbf{E}^\delta(x) \cdot \nabla\varphi\left(x, \frac{x}{\delta^q}, \frac{x}{\delta}\right) dx \\
&\quad = \int_\Omega \kappa_{1\delta}^2 \mathbf{E}^\delta(x)\left\{\delta\nabla_x\varphi(x,z,y) + \delta^{1-q}\nabla_z\varphi(x,z,y) + \nabla_y\varphi(x,z,y)\right\}\Big|_{z=\frac{x}{\delta^q},\, y=\frac{x}{\delta}} dx, \\
&\int_\Omega \delta \operatorname{curl}\mathbf{E}^\delta(x) \cdot \boldsymbol{\varphi}\left(x, \frac{x}{\delta^q}, \frac{x}{\delta}\right) dx \\
&\quad = \int_\Omega \mathbf{E}^\delta(x)\left\{\delta \operatorname{curl}_x \boldsymbol{\varphi}(x,z,y) + \delta^{1-q} \operatorname{curl}_z \boldsymbol{\varphi}(x,z,y) + \operatorname{curl}_y \boldsymbol{\varphi}(x,z,y)\right\}\Big|_{z=\frac{x}{\delta^q},\, y=\frac{x}{\delta}} dx, \\
&\int_\Omega \delta^q \operatorname{curl}\mathbf{E}^\delta(x) \cdot \boldsymbol{\varphi}_1\left(x, \frac{x}{\delta^q}, \frac{x}{\delta}\right) dx \\
&\quad = \int_\Omega \mathbf{E}^\delta(x)\left\{\delta^q \operatorname{curl}_x \boldsymbol{\varphi}_1(x,z,y) + \operatorname{curl}_z \boldsymbol{\varphi}_1(x,z,y)\right\}\Big|_{z=\frac{x}{\delta^q},\, y=\frac{x}{\delta}} dx.
\end{aligned}$$

Passing to the limit, as $\delta \to 0$, we obtain:

$$\iiint_{\Omega\times Z\times Y} \kappa_1^2(y)\mathbf{E}_0 \cdot \nabla_y\varphi \, dx\, dz\, dy = 0,$$

$$\iiint_{\Omega\times Z\times Y} \mathbf{E}_0 \cdot \operatorname{curl}_y\boldsymbol{\varphi} \, dx\, dz\, dy = 0, \tag{74}$$

$$\iiint_{\Omega\times Z\times Y} \boldsymbol{\chi} \cdot \boldsymbol{\varphi}_1 - \mathbf{E}_0 \cdot \operatorname{curl}_z \boldsymbol{\varphi}_1 \, dx\, dz\, dy = 0. \tag{75}$$

The equalities (74) imply that

$$\operatorname{div}_y\left(\kappa_1^2(y)\mathbf{E}_0\right) = 0, \qquad \operatorname{curl}_y \mathbf{E}_0 = 0. \tag{76}$$

It follows from $(76)_2$ [34] that there are a vector function $\mathbf{E}^1(x,z)$, $\mathbf{E}^1 \in L^2(\Omega \times Z)$, and scalar functions $v^k(y)$, $v^k \in H_{per}^1(Y)$, $k = 1,2,3$, such that

$$\begin{aligned}
&\mathbf{E}_0(x,z,y) = \mathbf{E}^1(x,z) + E_k^1(x,z)\nabla_y v^k(y), \\
&\mathbf{E}^1(x,z) = \frac{1}{|Y|}\int_Y \mathbf{E}_0(x,z,y)\, dy \equiv \widetilde{\mathbf{E}_0}.
\end{aligned} \tag{77}$$

According to the first equality in (76), the Y-periodic function $v^k(y)$ obeys the cell problem (70).

When the function $\boldsymbol{\varphi}_1$ in (75) does not depend on y we have:

$$\iint_{\Omega\times Z} \widetilde{\boldsymbol{\chi}}\boldsymbol{\varphi}_1 - \mathbf{E}^1 \operatorname{curl}_z \boldsymbol{\varphi}_1 \, dx\, dz = 0,$$

which implies that $\mathbf{E}^1 \in L^2(\Omega; H_{per}^1(\operatorname{curl}, Z))$ and $\operatorname{curl}_z, \mathbf{E}^1 = \widetilde{\boldsymbol{\chi}}$.

Let us take $\boldsymbol{\varphi}_1 = \nabla_y \xi_1(x,z,y)$ in (75), keeping in mind that $\operatorname{curl}_z \nabla_y \xi_1 = -\operatorname{curl}_y \nabla_z \xi_1$ and that the space $L^2_{sol,per}(Y)$ is orthogonal to $L^2_{pot,per}(Y)$ in $L^2(Y)$. Here

$$L^2_{sol,per}(Y) = \{\mathbf{u} \in \mathcal{C}^\infty_{per}(Y),\ \operatorname{div}\mathbf{u} = 0 \text{ in } Y\},$$
$$L^2_{pot,per}(Y) = \{\mathbf{u} \in L^2(Y),\ \mathbf{u} = \nabla w,\ \text{with } w \in H^1_{per}(Y)\}.$$

We obtain:

$$\iiint\limits_{\Omega\times Z\times Y} (\boldsymbol{\chi} - \operatorname{curl}_z \mathbf{E}^1)\cdot \nabla_y \xi_1 \, dx\, dz\, dy = -\iiint\limits_{\Omega\times Z\times Y} E^1_k(x,z)\nabla_y v^k \cdot \operatorname{curl}_y \nabla_z \xi_1 \, dx\, dz\, dy = 0.$$

Hence

$$\operatorname{div}_y(\boldsymbol{\chi} - \operatorname{curl}_z \mathbf{E}^1) = 0,$$

and there are Y-periodic vector functions $\mathbf{w}^k(y)$, $\mathbf{w}^k \in H^1_{per}(\mathrm{curl}, Y)$, $k = 1,2,3$, such that

$$\boldsymbol{\chi}(x,z,y) = \operatorname{curl}_z \mathbf{E}^1(x,z) + (\operatorname{curl}_z \mathbf{E}^1)_k(x,z)\operatorname{curl}_y \mathbf{w}^k(y).$$

Let us pass to the limit in equality (16) which can be written in the form,

$$\int\limits_\Omega \frac{\delta^{2q}\operatorname{curl}\mathbf{E}^\delta \cdot \operatorname{curl}\mathbf{u}^*}{\mu_\delta} - \kappa^2_{1\delta}\mathbf{E}^\delta \cdot \mathbf{u}^* - i4\pi\alpha_s^2\omega \mathbf{f}\cdot\mathbf{u}^*\, dx = 0. \tag{78}$$

Given test functions $\varphi(x,z)$ and $\varphi_1(x,z,y)$, with $\varphi \in \mathcal{D}(\Omega; \mathcal{C}^\infty_{per}(Z))$, $\varphi_1 \in \mathcal{D}(\Omega; \mathcal{C}^\infty_{per}(Z\times Y))$, we choose

$$\mathbf{u}^*(x) = \boldsymbol{\varphi}(x,z) + \delta^{1-q}\boldsymbol{\varphi}_1(x,z,y), \quad z = \frac{x}{\delta^q},\ y = \frac{x}{\delta},$$

and we pass to the limit in the corresponding equality (78) to obtain that

$$\iiint\limits_{\Omega\times Z\times Y} \frac{\boldsymbol{\chi}}{\mu(y)}(\operatorname{curl}_z \boldsymbol{\varphi} + \operatorname{curl}_y \boldsymbol{\varphi}_1) - \{\kappa_1^2(y)\mathbf{E}_0 + i4\pi\alpha_s^2\omega\mathbf{f}\}\boldsymbol{\varphi}\, dx\, dz\, dy = 0. \tag{79}$$

Taking $\boldsymbol{\varphi} = 0$ in (79) we find that

$$\operatorname{curl}_y\{\boldsymbol{\chi}(x,z,y)/\mu(y)\} = 0.$$

Hence, the Y-periodic function $\mathbf{w}^k(y)$, $k = 1,2,3$, solves the cell problem

$$\operatorname{curl}_y\left(\frac{\mathbf{e}^k + \operatorname{curl}_y \mathbf{w}^k}{\mu(y)}\right) = 0, \quad \widetilde{\mathbf{w}^k} = 0.$$

We introduce the constant homogenized matrix $(1/\mu)^h$ by:

$$\left(\frac{1}{\mu}\right)^h_{ik} = \frac{1}{|Y|}\int\limits_Y \frac{\delta_{ik} + (\operatorname{curl}_y \mathbf{w}^k)_i}{\mu(y)}\, dy.$$

Setting $\boldsymbol{\varphi}_1 = 0$ in (79) we derive the equation:

$$\operatorname{curl}_z\big((1/\mu)^h \operatorname{curl}_z \mathbf{E}^1(x,z)\big) - (\kappa_1^2)^h \mathbf{E}^1(x,z) = i4\pi\alpha_s^2\omega\mathbf{f}(x).$$

Finally, introducing the mean value,

$$\mathbf{E}(x) = \frac{1}{|Y||Z|}\iint\limits_{Z\times Y} \mathbf{E}_0(x,z,y)\, dz\, dy \equiv \frac{1}{|Z|}\int\limits_Z \mathbf{E}^1(x,z)\, dz, \tag{80}$$

we obtain that $\mathbf{E}^\delta \rightharpoonup \mathbf{E}$ weakly in $L^2(\Omega)$ and the function $\mathbf{E}$ solves the macroscopic equation (72).

It remains to show (73). We pass to the limit in the equality $\mathbf{D}^\delta = \varepsilon_\delta \mathbf{E}^\delta$ with help of the 3-scale convergence. We have,

$$\mathbf{D}^\delta \overset{3s}{\rightharpoonup} \mathbf{D}_0(x,z,y) = \varepsilon(y)\mathbf{E}_0(x,z,y),$$

then $\mathbf{D}^\delta \rightharpoonup \mathbf{D}$ weakly in $L^2(\Omega)$, with

$$\mathbf{D}(x) = \frac{1}{|Y||Z|} \iint\limits_{Z\times Y} \mathbf{D}_0(x,z,y)\,dz\,dy.$$

Using the representation formulas (77) and (80) we derive the relation,

$$\mathbf{D}(x) = E_k \frac{1}{|Y|} \int\limits_Y \varepsilon(y)\big(\mathbf{e}^k + \nabla_y v^k\big)\,dy,$$

hence $\mathbf{D} = \varepsilon^h \mathbf{E}$ where ε^h is the homogenized matrix defined by (71). Similarly, we derive the second relation in (73). The proof of Theorem 5 is achieved. □

6. Conclusion

We summarize here the principal novelties of the present study. We show that to give due consideration to different order relations between the lengths l, l_w, and l_s, one should study the microscopic time harmonic Maxwell equations with coefficients depending on δ. Remind that l is the representative periodicity cell size of the heterogeneous medium, l_w is the wave length, l_s is the skin layer length, L is the size of the domain of measurements Ω, with $\delta = l/L$ being a small dimensionless parameter. In contrast to the papers [4,22,47,49,50] on the differential operators arising in electromagnetism, we not only prove the two-scale or three-scale convergence theorems, as $\delta \to 0$, but find also the macroscopic equations resulting from homogenization of the Maxwell equations. Moreover, we derive boundary-value problems which allow for determination of the effective constants μ^h, ε^h, and σ^h; we comment also on the dispersion effect, i.e. the way how these constants and the macroscopic equations depend on the pulsation ω.

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